1) Bir parçacık $t=0$ anında,

$$\psi(x,t=0) = \begin{cases} A(a^2-x^2), & -a \leq x \leq a \\ 0, & \text{diğer yerlerde} \end{cases}$$

ile verilen dalga fonksiyonu ile temsil edilmektedir.

a) $\psi(x,0)$'ı normalleyiniz.

b) $t=0$ anında x'in beklenen değerini bulunuz.

c) $t=0$ anında P'nin beklenen değerini bulunuz.

d) x^2'nin beklenen değerini bulunuz.

e) p^2'nin beklenen değerini bulunuz.

a:

$$1 = \langle \psi(x,0) | \psi(x,0) \rangle = \int_{-\infty}^{+\infty} |\psi|^2 dx = \underbrace{\int_{-\infty}^{-a} |\psi|^2 dx}_{0} + \int_{-a}^{+a} |\psi|^2 dx + \underbrace{\int_{+a}^{\infty} |\psi|^2 dx}_{0}$$

$$1 = \int_{-a}^{+a} A^* (a^2-x^2)^* A (a^2-x^2)\, dx = |A|^2 \int_{-a}^{+a} (a^2-x^2)^2 dx = |A|^2 \int_{-a}^{+a} (a^4 - 2a^2x^2 + x^4)\, dx$$

$$\Rightarrow \frac{1}{|A|^2} = \left[a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right]_{-a}^{+a} \Rightarrow \frac{1}{|A|^2} = \left[a^5 - \frac{2a^5}{3} + \frac{a^5}{5} - a^5 - \frac{2a^5}{3} + \frac{a^5}{5} \right]$$

$$A = \sqrt{\frac{15}{16a^5}} \;//$$

b:

$$\langle x \rangle = \langle \psi | x | \psi \rangle = |A|^2 \int_{-a}^{+a} \underbrace{(a^2-x^2)^2 x}_{\text{tek fonk}}\, dx = A^2 \int_{-a}^{+a} (a^4 - 2a^2x^2 + x^4)\, x\, dx$$

$$\langle x \rangle = \underbrace{\frac{15}{16a^5}}_{|A|^2} \left[a^4 \frac{x^2}{2} - 2a^2 \frac{x^4}{4} + \frac{x^6}{6} \right]_{-a}^{+a} \Rightarrow \langle x \rangle = 0 \;//$$

c:

$$\langle P \rangle = \langle \psi | P | \psi \rangle = \langle \psi | \underbrace{\left(-i\hbar \frac{d}{dx}\right)}_{P_x} | \psi \rangle = |A|^2 \int_{-a}^{+a} (a^2-x^2) \left(-i\hbar \frac{d}{dx}\right) (a^2-x^2)\, dx$$

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$$\langle p \rangle = \frac{15}{16a^5}\int_{-a}^{+a}(a^2-x^2)\left[-i\hbar(-2x)\right]dx = \frac{15}{8a^5}\,i\hbar\int_{-a}^{+a}(a^2x-x^3)\,dx$$

$$\langle p \rangle = \frac{15i\hbar}{8a^5}\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_{-a}^{+a} = \frac{15i\hbar}{8a^5}\left[\cancel{\frac{a^4}{2}}-\cancel{\frac{a^4}{4}}-\left(\cancel{\frac{a^4}{2}}-\cancel{\frac{a^4}{4}}\right)\right] \Rightarrow \langle p \rangle = 0 \,//$$

d:
$$\langle x^2 \rangle = \langle \psi | x^2 | \psi \rangle = |A|^2\int_{-a}^{+a}(a^2-x^2)^2x^2\,dx = \frac{15}{16a^5}\int_{-a}^{+a}(a^4-2a^2x^2+x^4)\,x^2\,dx$$

$$\langle x^2 \rangle = \frac{15}{16a^5}\int_{-a}^{+a}\left[a^4x^2-2a^2x^4+x^6\right]dx = \frac{15}{16a^5}\left[a^4\frac{x^3}{3}-2a^2\frac{x^5}{5}+\frac{x^7}{7}\right]_{-a}^{+a}$$

$$\langle x^2 \rangle = \frac{15}{16a^5}\left[\left(\frac{a^7}{3}-\frac{2a^7}{5}+\frac{a^7}{7}\right)-\left(-\frac{a^7}{3}+\frac{2a^7}{5}-\frac{a^7}{7}\right)\right] = \frac{15}{16a^5}\,2\left(\frac{a^7}{3}+\frac{2a^7}{5}-\frac{a^7}{7}\right)$$

$$\langle x^2 \rangle = \frac{a^2}{7}\,//$$

e:
$$\langle p^2 \rangle = \langle \psi | p^2 | \psi \rangle = \langle \psi | \left(-\hbar^2\frac{\partial^2}{\partial x^2}\right) | \psi \rangle = |A|^2\int_{-a}^{+a}(a^2-x^2)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)(a^2-x^2)\,dx$$

$$\langle p^2 \rangle = \frac{-15\hbar^2}{16a^5}\int_{-a}^{+a}(a^2-x^2)\frac{d}{dx}(-2x)\,dx = \frac{15\hbar^2}{8a^5}\int_{-a}^{+a}(a^2-x^2)(1)\,dx$$

$$\langle p^2 \rangle = \frac{15\hbar^2}{8a^5}\left[a^2x-\frac{x^3}{3}\right]_{-a}^{+a} = \frac{15\hbar^2}{8a^5}\left[a^3-\frac{a^3}{3}-\left(-a^3+\frac{a^3}{3}\right)\right] = \frac{15\hbar^2}{4a^5}\left(a^3-\frac{a^3}{3}\right)$$

$$\langle p^2 \rangle = \frac{15\hbar^2}{2a^5}\,\frac{a^3}{3} \Rightarrow \langle p^2 \rangle = \frac{5\hbar^2}{2a^2}\,//$$

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2) Küresel simetriye sahip bir $V(r)$ potansiyeli etkisi altında hareket eden bir taneciğin dalga fonksiyonu, A ve α birer sabit olmak üzere, $\psi(r) = A e^{-\alpha r^2}$ ile verilmektedir. $V(r)$ potansiyelini ve taneciğin toplam enerjisini bir sabit farkı ile belirleyiniz.

$$H|\psi(r)\rangle = E|\psi(r)\rangle \Rightarrow \left[\frac{p^2}{2m} + V(r)\right]|\psi(r)\rangle = E|\psi(r)\rangle$$

$$p = -i\hbar\vec{\nabla} \Rightarrow p^2 = -\hbar^2\nabla^2 = -\hbar^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$\psi(\underline{\underline{r}}) \Rightarrow \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + 0 + 0$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r)}{\partial r}\right)\right] + V(r)\psi(r) = E\psi(r)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A(-\alpha 2r\, e^{-\alpha r^2})\right)\right] + (V(r)-E)\psi(r) = 0$$

$$\Rightarrow \frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(-A\alpha 2r^3 e^{-\alpha r^2}\right)\right] + (E-V(r))\psi(r) = 0$$

$$\Rightarrow \frac{\hbar^2}{2m}\left[\frac{1}{r^2}\left(-A\alpha 6r^2 e^{-\alpha r^2} - A\alpha 2r^3(-2\alpha r)e^{-\alpha r^2}\right)\right] + (E-V(r))\cdot\psi = 0$$

$$\Rightarrow -\frac{\hbar^2 A\alpha}{mr}\left[3r e^{-\alpha r^2} - 2r^3\alpha e^{-\alpha r^2}\right] + (E-V(r))\psi = 0$$

$$\Rightarrow -\frac{\hbar^2\alpha}{m\cancel{r}}\left[3\cancel{r}\underbrace{Ae^{-\alpha r^2}}_{\psi} - 2\overset{2}{\cancel{r^3}}\alpha\underbrace{Ae^{-\alpha r^2}}_{\psi}\right] + (E-V(r))\psi = 0$$

$$\Rightarrow \frac{\hbar^2}{2m}\left[-2\alpha(3-2\alpha r^2)\right] + E - V = 0$$

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$$\Rightarrow V(r) = E - \frac{\hbar^2 \alpha}{m}\left(3 - 2\alpha r^2\right)$$

$$V(r) = \frac{2\alpha^2\hbar^2}{m} r^2 \Big(\underbrace{E - \frac{3\hbar^2\alpha}{m}}_{\text{sabit} = c} \Big) \qquad E = c + \frac{3\hbar^2\alpha}{m}$$

3) $\psi(x,y,z) = A e^{-1/2\,(x^2+y^2+z^2)}$ dalga fonksiyonu ile temsil edilen taneciğin R yarıçaplı bir kürede bulunma olasılığını hesaplayınız.

$\psi(x,y,z)$'yi normalizyalım;

$$1 = \langle\psi|\psi\rangle = \int_{x=-\infty}^{+\infty}\int_{y=-\infty}^{+\infty}\int_{z=-\infty}^{+\infty} |A|^2 e^{-(x^2+y^2+z^2)} \underbrace{dx\,dy\,dz}_{dV}$$

$$\frac{1}{|A|^2} = \int_{-\infty}^{+\infty} e^{-x^2}dx \int_{-\infty}^{+\infty} e^{-y^2}dy \int_{-\infty}^{+\infty} e^{-z^2}dz \qquad \int_0^{\infty} x^{2n} e^{-x^2/\alpha^2}dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{\alpha}{2}\right)^{2n+1}$$

$$I = \int_{-\infty}^{+\infty} e^{-x^2}dx = 2\int_0^{\infty} e^{-x^2}dx = 2\sqrt{\pi}\,\frac{(2.0)!}{0!}\left(\frac{1}{2}\right)^{2.0+1} = \sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-y^2}dy = \int_{-\infty}^{+\infty} e^{-z^2}dz$$

$n=0$

$\alpha=1$

$$\Rightarrow \frac{1}{|A|^2} = \sqrt{\pi}\sqrt{\pi}\sqrt{\pi} \Rightarrow A = \pi^{-3/4}$$

veya, küresel koordinatlara dönüştürerek;

$$\psi(x,y,z) \longrightarrow \psi(r,\theta,\phi) = A e^{-r^2/2}, \qquad x^2+y^2+z^2 = r^2$$

$$\Rightarrow 1 = \langle\psi|\psi\rangle = |A|^2 \int_0^{\infty} e^{-r^2} r^2 \sin\theta\, dr\, d\theta\, d\phi = |A|^2 \int_0^{\infty} r^2 e^{-r^2} dr \underbrace{\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}$$

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$$\Rightarrow \frac{1}{|A|^2} = 4\pi \int_0^{\infty} r^2 e^{-r^2} dr = 4\pi \sqrt{\pi} \left(\frac{1}{2}\right)^{2\cdot 1+1} \frac{(2\cdot 1)!}{1!} = 4\pi\sqrt{\pi}\,\frac{1}{8}\cdot 2$$

$n=1$

$\alpha=1$

$$A = \pi^{-3/4} \;//$$

$$\bullet \; P(0<r<R) = |A|^2 \int_V e^{-r^2} r^2 \sin\theta \, dr\, d\theta\, d\phi = \pi^{-3/2} \int_0^R r^2 e^{-r^2} dr \underbrace{\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}$$

$$P(0<r<R) = 4\pi^{-1/2} \int_0^R r^2 e^{-r^2} dr$$

$u = r \qquad dv = r e^{-r^2} dr \qquad \int u\, dv = uv - \int v\, du$

$du = dr \qquad v = -\frac{e^{-r^2}}{2}$

$$P(0<r<R) = \frac{4}{\sqrt{\pi}} \left[r\left(-\frac{e^{-r^2}}{2}\right)\Bigg|_0^R + \int_0^R \frac{+e^{-r^2}}{2} dr \right] = \frac{4}{\sqrt{\pi}} \left[-\frac{Re^{-R^2}}{2} + 0 + \int_0^R \frac{e^{-r^2}}{2} dr \right]$$

$$P(0<r<R) = \frac{4}{\sqrt{\pi}} \left[\frac{-Re^{-R^2}}{2} + \int_0^R \frac{e^{-r^2}}{2} dr \right]$$

$\int_0^R e^{-r^2} dr \Big\}$ seriye açarak çözülür...

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