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Ph.D. THESIS

SOME IDENTIFICATION PROBLEMS FOR SEMILINEAR PARABOLIC EQUATIONS

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LIST OF SYMBOLS AND ABBREVIATIONS

Symbol	Explanation
$\mathbb{R}^n$	: n-Dimensional Euclidean Space
H	: Hilbert Space
E	: Banach Space
T_i	: Linear Operator
T^*	: Hilbert Adjoint of the Linear Operator T
A	: Positive Definite Self-Adjoint Operator
$D(A)$	: Domain of the Operator A
$C((0,T),E)$	: Space of all Continuous Functions on $(0,T)$ with values in E
α	: Contractivity Factor
K_i	: Positive Constant

Abbreviation	Explanation
FDM	: Finite Difference Method
PDE	: Partial Differential Equation
SIP	: Source Identification Problem

SUMMARY

Ph.D. THESIS

SOME IDENTIFICATION PROBLEMS FOR SEMILINEAR PARABOLIC EQUATIONS

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In this thesis, a time- and a space-dependent source identification problems governed by semilinear differential equations are considered. For solving these problems numerically the first order of accuracy stable Rothe difference schemes are proposed. Results for existence and uniqueness of the solutions of the differential problems and the proposed difference schemes are established applying Banach fixed-point theorem. Furthermore, the proposed difference schemes are tested on examples to demonstrate their efficiency and also some numerical discussions are given. Finally, a summary and some concluding remarks are presented.

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Keywords: Identification problems, inverse problems, semilinear equation, existence and uniqueness, numerical solution.

ÖZET

DOKTORA TEZİ

YARILİNEER PARABOLİK DENKLEMLER İÇİN BAZI ÖZDEŞLEŞTİRME PROBLEMLERİ

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Bu tezde, yarılineer diferensiyel denklemler için bir zamana bağlı ve bir uzaya bağlı kaynak belirleme problemi ele alınmıştır. Bu problemleri sayısal olarak çözmek için birinci mertebeden, kararlı Rothe fark şemaları önerilmiştir. Banach sabit nokta teoremi kullanılarak bu diferensiyel problemlerin ve önerilen fark şemalarının çözümlerinin varlık ve teklik sonuçları ortaya konmuştur. Ayrıca, önerilen fark şemalarının etkinliğini göstermek için bu fark şemaları örnekler üzerinde test edilmiş ve bazı sayısal tartışmalar verilmiştir. Son olarak özet ve bazı tamamlayıcı notlar verilmiştir.

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Anahtar kelimeler: Özdeşleştirme problemleri, ters problemler, yarılineer denklem, varlık ve teklik, sayısal çözüm.

1. INTRODUCTION

It is known that the mathematical models of real-life problems play a crucial role to give an idea for understanding the natural world. For this reason, considerably many researchers have tried and successed modeling the real-life problems in science and engineering using partial differential equations (PDEs), because they can describe many phenomena such as diffusion, heat conduction, population dynamics, biomedical processes etc. Until about the last three decades, the researchers have utilized the mathematical models consisting the direct problems. For example, in [1] a dynamical model, involving a direct problem for a PDE, for the growth and size distribution of multiple metastatic tumors is presented. As days go by, more complicated and more comprehensive models have been started to be considered with the advancements in science and technology. In these new models, the researchers have used the inverse problems very often. That is why, a growing interest has been established for the study of these problems. For example, in [2], an inverse problem that models an actual problem in thermochronology, a branch of geology wherein the thermal history of minerals is evaluated, is considered. Moreover, in [3], an inverse problem arising in a model for a physiologically structured population is investigated.

In inverse problems one is not only aim to find the solution but also the missing parameter (coefficient/condition/source) in the problem. For this reason, inverse problems are generally viewed in the main classes of coefficient, condition or source identification problems. In [4], these main classes are characterized in detail and some examples are illustrated. Actually, the missing parameter in an inverse problem may be considered as a control parameter. Another reason for the growing interest to inverse problems is the need for the control of systems that are modeled.

The coefficient identification problems have been studied extensively by many researchers [5–9]. In [5] and [6], the authors have identified the time-dependent coefficient of heat capacity and the diffusion coefficient, respectively. In [7], the total set of coefficients of an inhomogeneous quasilinear heat-conduction equation is determined. In [9], an identification problem of the leading time-dependent coefficient is studied. Paper [9] is devoted to review the Carleman estimates that are very significant in the theory of coefficient identification problems.

The condition identification problems have taken less interest by the researchers comparing the coefficient problems. Murio has identified the boundary condition(s) applying the mollification method in papers [10, 11]. Yaparova obtained the unknown initial condition applying a numerical method based on finite difference schemes and a regularization technique in [12].

The source identification problems (SIPs), because of their importance in the models, are the most frequently encountered inverse problems. In SIPs, the unknown source term may be only time- or space-dependent, or time and space-dependent simultaneously. Several time- and space-dependent SIPs are investigated from different aspects in papers [13–20] and [21–25], respectively.

For identifying the unknown paramater we require an additional measurement or, mathematically, an additional condition that is called as overdetermination. The overdetermination may be given in the form of an internal measurement in the domain [13, 15, 17] or an integral constraint [6, 16, 26]. Of course, depending on the form of additional condition (overdetermination) the techniques (or tools) that are used in the investigations differ.

Generally speaking, by exact solution of an inverse problem one means a solution satisfying the governing equation and initial/boundary/overdetermined conditions. Several methods for obtaining the solutions of inverse problems can be found in [4, 27–29]. Unfortunately, due to the nonlinear terms and/or variable coefficients in a number of these problems it is very difficult to obtain their exact solutions. At this point, with the contributions that the advancements in computer science have made, considerably many investigations are conducted for the numerical solutions of these problems [9–17]. For example, Borukhov [30] used a numerical method, in which the problem is turned into the Volterra integral equation, for solving an inverse problem,. Similarly, Yaparova [31] used a numerical method for solving a condition identification problem, that is based on reducing the problem to an integral equation by the Laplace transform. Yang *et al* [32] developed a numerical procedure involving the Landweber iteration to solve a SIP. Another numerical approach for solving SIPs involves the utilization of the schemes of finite difference method (FDM) [13, 15–17, 33, 34].

In the literature, the SIPs for linear equations are of greater interest comparing for

the nonlinear ones. For example, Kamynin [35] studied the unique solvability of a space-dependent SIP for linear parabolic equation. Guidetti [26] established the existence and uniqueness results for the solution, which is also of maximal regularity type, of a SIP in an abstract parabolic system. Orazov and Sadybekov [22] investigated a class of SIPs from unique solvability point of view.

In our previous studies, we have investigated several SIPs for linear PDEs from different aspects in [16, 17]. On the other hand, after a literature review, (to the best of our knowledge) we noticed that SIPs for semilinear PDEs have not been considered before. In fact, we aimed to fill this gap with this thesis.

We arrange the sequel of the thesis in the following way: In Section 2, the preliminaries and a mathematical background is given. In Section 3, a time-dependent SIP for semilinear parabolic equations subject to an integral overdetermination is investigated. In Section 4, an abstract SIP for a semilinear differential equation subject to a final overdetermination, that has an application as a space-dependent SIP for a semilinear parabolic equation is investigated.

2. MATERIALS AND METHODS

2.1. NORMED SPACES

Definition 2.1.1. [36, Definition 2.2-1] Given that X is a vector space over the field $\mathscr{F}$ ($\mathbb{C}$ or $\mathbb{R}$). A norm on X is a function $\|\cdot\| : X \to \mathscr{F}$ such that

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = \overline{0} \in X$

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

for any $x, y \in X$ and for any $\lambda \in \mathscr{F}$. The vector space X with a norm defined on it is called a normed space and is denoted by $(X, \|\cdot\|)$.

For example, $(\mathbb{R}^n, \|\cdot\|)$, $\left(C[a,b], \|\cdot\|_p\right)$ and $\left(L_p[a,b], \|\cdot\|_p\right)$, $1 \leq p < \infty$ are some normed spaces.

Definition 2.1.2. [36, Definition 2.4-4] Given that $\|\cdot\|$ and $\|\cdot\|_0$ are two norms on a normed space X. The norms $\|\cdot\|$ and $\|\cdot\|_0$ are called as equivalent, if there exist $a, b > 0$ such that

$$a \|x\|_0 \leq \|x\| \leq b \|x\|_0$$

for any $x \in X$.

Example 2.1.1. On the normed space $X = C[a,b]$, the norms $\|x\|_0 = \max\limits_{0 \leq t \leq T} |x(t)|$ and $\|x\| = \max\limits_{0 \leq t \leq T} e^{-\mu t} \, |x(t)|$, $\mu > 0$ are equivalent.

Solution: Since

$$e^{-\mu T} \leq e^{-\mu t} \leq 1$$

for some fixed $\mu > 0$, we have

$$e^{-\mu T} \|x\|_0 \leq \|x\| \leq \|x\|_0,$$

and so these norms are equivalent. ■

Example 2.1.2. On the normed space $X = C[a,b]_\tau$, the norms $\|x\|_* = \max\limits_{0\le k\le N} |x(t_k)|$ and $\|x\|_{**} = \max\limits_{0\le k\le N} \frac{1}{(1+\mu\tau)^k} |x(t_k)|$, $\mu > 0$ are equivalent.

Solution: Since

$$\frac{1}{(1+\mu\tau)^N} \le \frac{1}{(1+\mu\tau)^k} \le 1$$

for all $0 \le k \le N$, we have

$$\frac{1}{(1+\mu\tau)^N} \|x\|_* \le \|x\|_{**} \le \|x\|_* .$$

Thus, these norms are equivalent. ■

Definition 2.1.3. Assume that $\frac{1}{p} + \frac{1}{q} = 1$ with $p,q > 1$ and $(X, \|\cdot\|)$ is a normed space. Then, for any $f,g \in X$, Hölder inequality states that

$$\int_a^b \|f(x)\,g(x)\|\,dx \le \left\{\int_a^b \|f(x)\|^p\,dx\right\}^{1/p} \left\{\int_a^b \|g(x)\|^q\,dx\right\}^{1/q}.$$

In the case when $p = q = 2$, this inequality becomes

$$\int_a^b \|f(x)\,g(x)\|\,dx \le \left\{\int_a^b \|f(x)\|^2\,dx\right\}^{1/2} \left\{\int_a^b \|g(x)\|^2\,dx\right\}^{1/2},$$

and is called as Cauchy-Schwarz inequality.

The discrete analogues of these inequalities (for sum) are

$$\sum_{k=1}^N \|f_k g_k\| \le \left\{\sum_{k=1}^N \|f_k\|^p\right\}^{1/p} \left\{\sum_{k=1}^N \|g_k\|^q\right\}^{1/q},$$

and

$$\sum_{k=1}^N \|f_k g_k\| \le \left\{\sum_{k=1}^N \|f_k\|^2\right\}^{1/2} \left\{\sum_{k=1}^N \|g_k\|^2\right\}^{1/2},$$

respectively.

Definition 2.1.4. Let $(X, \|\cdot\|)$ be a normed space and let $(x_n) \in X$. The sequence (x_n) is said to be a Cauchy sequence in X, if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \ \forall n, m \geq N,$$

or simply

$$\|x_n - x_m\| \overset{n,m\to\infty}{\longrightarrow} 0 \ \forall n, m \geq N.$$

Corollary 2.1.1. Every convergent sequence is also a Cauchy sequence, but the converse need not be true.

Solution: Assume that (x_n) is a convergent sequence in a normed space X, then there exists $N \in \mathbb{N}$ such that

$$\|x_n - x\| \overset{n\to\infty}{\longrightarrow} 0 \ \forall n \geq N.$$

Then, applying the triangle inequality, we get

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| \overset{n,m\to\infty}{\longrightarrow} 0 \ \forall n, m \geq N.$$

Conversely, one may consider the following examples:

1) Let us take the sequence $(f_n(t)) = \{t^n\}$ from the normed space $(C[0,1], \|\cdot\|_1)$. Obviously, (without loss of generality) assuming $n \leq m$, we have that

$$\begin{aligned} \|f_n(t) - f_m(t)\| &= \int_0^1 |t^n - t^m| \, dt = \int_0^1 (t^n - t^m) \, dt \\ &= \frac{1}{n+1} - \frac{1}{m+1} \overset{n,m\to\infty}{\longrightarrow} 0, \end{aligned}$$

and so (f_n) is a Cauchy sequence. However, since

$$f_n(t) \overset{n\to\infty}{\longrightarrow} f(t) = \begin{cases} 1, \text{ if } t = 1 \\ 0, \text{ if } t \neq 1 \end{cases},$$

which is not an element in $C[0,1]$, and so (f_n) is not a convergent sequence in $C[0,1]$.

2) The sequence $(p_n(x)) = \left\{\frac{x^n}{n!}\right\} \in P[a,b]$, that is the space of all polynomials defined on $[a,b]$, is a Cauchy sequence but since

$$p_n(x) \overset{n\to\infty}{\longrightarrow} e^x \notin P[a,b],$$

(p_n) is not a convergent sequence in $P[a,b]$. ■

Definition 2.1.5. Given that $(X, \|\cdot\|)$ is a normed space, then X is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.1.6. A complete normed space is called a Banach space. In other words, Banach space is a normed space in which every Cauchy sequence is convergent.

Example 2.1.3. The normed spaces $(\mathbb{R}^n, \|\cdot\|)$, $(C[a,b], \|\cdot\|_\infty)$ and $\left(L_p[a,b], \|\cdot\|_p\right)$ for $1 \leq p < \infty$ are some examples for Banach spaces. On the other hand, the normed space $X = C_p[0,1]$, $1 \leq p < \infty$ is not a Banach space with respect to the norm $\|\cdot\|_p$.

Corollary 2.1.2. Obviously, while a normed space is not a Banach space with respect to a norm, it may be a Banach space with respect to another norm. For example, the space $X = C[0,1]$ is not a Banach space with respect to the norm $\|\cdot\|_1$, but it is a Banach space with respect to the norm $\|\cdot\|_\infty$.

Definition 2.1.7. [36, Definition 3.1-1] Given that X is a vector space over the field $\mathscr{F}$ ($\mathbb{C}$ or $\mathbb{R}$). An inner product on X is a function $\langle\cdot,\cdot\rangle : X \times X \to \mathscr{F}$ such that

(i) $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ if and only if $x = \overline{0} \in X$

(ii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$

(iii) $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$

(iv) $\langle \lambda x,y\rangle = \lambda \langle x,y\rangle$

for any $x,y,z \in X$ and for any $\lambda \in \mathscr{F}$. The space X with an inner product defined on it is called an inner product space.

Remark 2.1.1. Any inner product on a normed space X defines a norm with

$$\langle x,x\rangle = \|x\|^2.$$

Remark 2.1.2. Given that $(X, \|\cdot\|)$ is a normed space. The space X is an inner product space if and only if the norm $\|\cdot\|$ satisfies the parallelogram law, that is

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \ \forall x, y \in X. \tag{2.1}$$

Definition 2.1.8. A complete inner product space is called a Hilbert space.

Example 2.1.4. The normed spaces $(\mathbb{R}^n, \|\cdot\|)$ and $(L_2[a,b], \|\cdot\|_2)$ are some examples for Hilbert spaces.

Example 2.1.5. The following normed spaces are not Hilbert spaces:

1) The normed space $\left(C_p[0,1], \|\cdot\|_p\right)$, $1 \leq p < \infty$ is not a Hilbert space, because it is not complete (sf. Example 2.1.3).

2) The normed space $(C[0,1], \|\cdot\|_\infty)$ is not a Hilbert space, because eventhough it is complete, it is not an inner product space. To show this, let us take two elements $f(t) = 1$, $g(t) = t \in C[0,1]$. Then, since

$$\begin{aligned} \|f+g\|_\infty &= \max_{0\leq t\leq 1} |t+1| = 2, \\ \|f-g\|_\infty &= \max_{0\leq t\leq 1} |1-t| = 1, \\ \|f\|_\infty &= 1 = \|g\|_\infty, \end{aligned}$$

obviously the norm does not satisfy the parallelogram law given in (2.1).

3) The normed space $\left(L_p[0,1], \|\cdot\|_p\right)$, $1 \leq p < \infty$ with $p \neq 2$ is not a Hilbert space. Similar to previous example, eventhough it is complete, it is not an inner product space.

Corollary 2.1.3. Since every inner product defines a norm on X, inner product spaces are also normed spaces. Moreover, Hilbert spaces are also Banach spaces.

2.2. LINEAR OPERATORS IN NORMED SPACES

Definition 2.2.1. [36, Definition 2.6-1] Given that X, Y are two vector spaces. An operator

$T : X \to Y$ is called as linear, if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \ \forall x, y \in X, \ \forall \alpha, \beta \in \mathbb{R}.$$

The null space of the operator T is defined by

$$Null\ (T) = \{x \in X : T(x) = 0\}.$$

Example 2.2.1. The following operators are linear:

1) The Fredholm integral operator T_1 from $C[a,b]$ into itself given by

$$T_1 f = \int_a^b f(t)\, dt.$$

2) The differentiation operator T_2 from $C^{(2)}(a,b)$ into $C(a,b)$ given by

$$T_2 f = -a(t) f''(t) + \sigma f(t), \ \sigma \geq 0.$$

3) The differentiation operator T_3 from $C^{(2)}(a,b)$ into $C(a,b)$ given by

$$T_3 f = -\left(a(t) f'(t)\right)' + \sigma f(t), \ \sigma \geq 0.$$

Solution: Let us show that the given operators are linear.

1) Since

$$\begin{aligned} T_1(\alpha f + \beta g)(t) &= \int_a^b (\alpha f + \beta g)(t)\, dt = \alpha \int_a^b f(t)\, dt + \beta \int_a^b g(t)\, dt \\ &= \alpha T_1 f + \beta T_1 g, \end{aligned}$$

the operator T_1 is linear.

2) Since

$$\begin{aligned} T_2(\alpha f+\beta g)(t) &= -a(t)(\alpha f+\beta g)''(t)+\sigma(\alpha f+\beta g)(t) \\ &= \alpha\left[-a(t)f''(t)+\sigma f(t)\right]+\beta\left[-a(t)g''(t)+\sigma g(t)\right] \\ &= \alpha T_2 f(t)+\beta T_2 g(t), \end{aligned}$$

the operator T_2 is linear.

3) Since

$$\begin{aligned} T_3(\alpha f+\beta g)(t) &= \left(-a(t)(\alpha f+\beta g)'(t)\right)'+\sigma(\alpha f+\beta g)(t) \\ &= \alpha\left[-\left(a(t)f'(t)\right)'+\sigma f(t)\right]+\beta\left[-\left(a(t)g'(t)\right)'+\sigma g(t)\right] \\ &= \alpha T_3 f+\beta T_3 g, \end{aligned}$$

the operator T_3 is linear. ■

Theorem 2.2.1. [36, Theorem 2.6-10] Given that X, Y are two vector spaces over the field $\mathscr{F}$ ($\mathbb{C}$ or $\mathbb{R}$) and that $T: D(T) \to Y$ is a linear operator with domain $D(T) \subset X$ and range $R(T) \subset Y$. Then,

(i) the inverse of T exists if and only if $Null\ (T) = \{0 \in X\}$;

(ii) if T^{-1} exists, it is linear, as well.

Definition 2.2.2. [36, Definition 2.7-1] Given that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed spaces and $T: X \to Y$ is a linear operator. The operator T is said to be bounded if there is a real number $K > 0$ such that

$$\|Tx\|_Y < K\|x\|_X \ \forall x \in D(T).$$

Moreover, the norm of the operator T is defined by

$$\begin{aligned} \|T\| &= \inf\{K > 0 : \|Tx\|_Y < K\|x\|_X \ \forall x \in D(T)\} \\ &:= \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}. \end{aligned}$$

Example 2.2.2. The following operators are bounded:

1) The Fredholm integral operator $T_1 : (C[a,b], \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$ defined in *1) of* Example 2.2.1.

2) The multiplication operator T_4 from $(C[0,1], \|\cdot\|_\infty)$ into itself, given by

$$T_4 f = \left(t^3 + 5\right) f(t).$$

Solution: Let us show that the given operators are bounded.

1) Since T_1 is linear (sf. Example 2.2.1) and

$$\begin{aligned} |T_1 f| &= \left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt \leq \max_{a\leq t\leq b} |f(t)| \int_a^b dt \\ &= (b-a) \max_{a\leq t\leq b} |f(t)| = (b-a) \|f\|_\infty, \end{aligned}$$

the operator T_1 is bounded and bounded by $K = b - a$.

2) Obviously T_4 is linear, and since

$$\|T_4 f\|_\infty = \max_{0\leq t\leq 1} \left| \left(t^3 + 5\right) f(t) \right| \leq 6 \max_{0\leq t\leq 1} |f(t)| = 6 \|f\|_\infty,$$

the operator T_4 is bounded and bounded by $K = 6$. ■

Example 2.2.3. The differential operator $T_5 f = f'(t)$ is an unbounded operator in $\left(C^{(2)}[0,1], \|\cdot\|_\infty\right)$.

Solution: Clearly, T_5 is linear. Let us take the function $f_n(t) = t^n$ for fixed n, then since

$$\frac{\|T_5 f_n\|_\infty}{\|f_n\|_\infty} = \frac{\max\limits_{0\leq t\leq 1} \left|nt^{n-1}\right|}{\max\limits_{0\leq t\leq 1} |t^n|} = n \xrightarrow{n\to\infty} \infty,$$

the operator T_5 is unbounded. ■

Similarly, the differentiation operators $T_2 : \left(C^{(2)}[0,1], \|\cdot\|_\infty\right) \to (C[0,1], \|\cdot\|_\infty)$ and $T_3 : \left(C^{(2)}[0,1], \|\cdot\|_2\right) \to (C[0,1], \|\cdot\|_2)$ defined in *2) and 3) of* Example 2.2.1, respectively, are unbounded.

2.3. ADJOINT OF AN OPERATOR

Definition 2.3.1. [37, p. 19] Given that T is a linear operator from a Hilbert space H_1 into another one H_2. Then, the Hilbert adjoint operator T^* of T is the operator

$$T^* : H_2 \to H_1$$

such that for any $x \in H_1$ and $y \in H_2$

$$\langle Tx, y\rangle = \langle x, T^* y\rangle .$$

Theorem 2.3.1. The Hilbert adjoint operator T^* of T is linear and unique.

Definition 2.3.2. [37, pp. 19-20] A linear operator $T : H \to H$ is called self-adjoint, if $\langle Tx, y\rangle = \langle x, Ty\rangle$, i.e., $T^* = T$ for any $x, y \in H$; and is called positive definite in H, if

$$\langle Tx, x\rangle \geq \delta \langle x, x\rangle , \ \delta \geq \delta_0 > 0$$

for all nonzero x in H.

Example 2.3.1. The differentiation operator T_3 defined by

$$T_3 f = -\left(a(x) f'(x)\right)' + \sigma f(x),$$

where $a(x) \geq a > 0$ and $\sigma \geq 0$, with the domain

$$D(T_3) = \left\{ f, f', f'' \in L_2[0, L] : f'(0) = f(L) = 0 \right\}$$

is self-adjoint and positive definite in $L_2[0, L]$.

Solution: Recall that T_3 is linear (sf. Example 2.2.1). For all $f, g \in D(T_3)$ and $\sigma \geq 0$, we have

$$\begin{aligned} \langle T_3 f, g\rangle &= \int_0^L \left[-\left(a(x) f'(x)\right)' + \sigma f(x) \right] g(x)\, dx \\ &= -\int_0^L \left(a(x) f'(x)\right)' g(x)\, dx + \sigma \int_0^L f(x)\, g(x)\, dx. \end{aligned}$$

Applying integration by parts, we get

$$\begin{aligned}\langle T_3 f, g\rangle &= -a(x) f'(x) g(x)\big]_{x=0}^{L} + \int_0^L a(x) f'(x) g'(x)\, dx + \sigma \int_0^L f(x) g(x)\, dx \\ &= \int_0^L a(x) f'(x) g'(x)\, dx + \sigma \int_0^L f(x) g(x)\, dx. \qquad (2.2)\end{aligned}$$

Applying integration by parts once again, we reach

$$\begin{aligned}\langle T_3 f, g\rangle &= a(x) f(x) g'(x)\big]_{x=0}^{L} - \int_0^L f(x) \left(a(x) g'(x)\right)' dx + \sigma \int_0^L f(x) g(x)\, dx \\ &= -\int_0^L f(x) \left(a(x) g'(x)\right)' dx + \sigma \int_0^L f(x) g(x)\, dx \\ &= \int_0^L f(x) \left[-\left(a(x) g'(x)\right)' + \sigma g(x)\right] dx \\ &= \langle f, T_3 g\rangle,\end{aligned}$$

which implies that $T_3^* = T_3$. Hence, T_3 is a self-adjoint operator in $L_2[0,L]$.

Now, we shall show that T_3 is positive definite. The cases $\sigma > 0$ and $\sigma = 0$ will be considered seperately. For all nonzero $f \in D(T_3)$ and $\sigma > 0$, from (2.2) we have

$$\langle T_3 f, f\rangle = \int_0^L a(x) \left(f'(x)\right)^2 dx + \sigma \langle f, f\rangle.$$

Taking into account that $a(x) \geq a > 0$, we obtain

$$\langle T_3 f, f\rangle \geq \sigma \langle f, f\rangle,$$

which means that T_3 is positive definite in $L_2[0,L]$ with $\delta = \sigma > 0$.

Similarly for all nonzero $f \in D(T_3)$ and $\sigma = 0$, taking into account that $a(x) \geq a > 0$, from (2.2) we have

$$\langle T_3 f, f\rangle = \int_0^L a(x) \left(f'(x)\right)^2 dx \geq a \int_0^L \left(f'(x)\right)^2 dx.$$

Now, all we need to do is to prove that

$$\int_0^L \left(f'(x)\right)^2 dx \geq \int_0^L f^2(x)\, dx = \langle f, f\rangle.$$

To show this, since $f(L)=0$, we start with

$$f(x)=-\int_x^L f'(s)\,ds.$$

Denoting $s=x-y+L$, we get

$$f(x)=-\int_x^L f'(x-y+L)\,dy=-\int_0^L \phi(x-y+L)\,dy,$$

where

$$\phi(x-y+L)=\begin{cases} f'(x-y+L), & 0\le x\le y\le L,\\ 0, & 0\le y\le x\le L.\end{cases}$$

Then, to change the order of integrals applying the Minkowki's inequality, we have

$$\begin{aligned}\|f\|_H &= \sqrt{\langle f,f\rangle}=\left\{\int_0^L f^2(x)\,dx\right\}^{1/2}\\ &= \left\{\int_0^L\left[-\int_0^L \phi(x-y+L)\,dy\right]^2 dx\right\}^{1/2}\\ &\le \int_0^L\left[\int_0^L \phi^2(x-y+L)\,dx\right]^{1/2} dy\\ &= \int_0^L\left[\int_0^y \phi^2(x-y+L)\,dx+\int_y^L \phi^2(x-y+L)\,dx\right]^{1/2} dy\\ &= \int_0^L\left\{\int_0^y \phi^2(x-y+L)\,dx\right\}^{1/2} dy.\end{aligned}$$

Now, denoting back $x-y+L=s$, we deduce

$$\begin{aligned}\sqrt{\langle f,f\rangle} &\le \int_0^L\left[\int_{L-y}^L \left[f'(s)\right]^2 ds\right]^{1/2} dy \le \int_0^L\left[\int_0^L \left[f'(s)\right]^2 ds\right]^{1/2} dy\\ &= L\sqrt{\langle f',f'\rangle}.\end{aligned}$$

From that it follows

$$\int_0^L \left[f'(x)\right]^2 dx \ge \frac{1}{L^2}\langle f,f\rangle.$$

Consequently, for $\sigma=0$, $\langle T_3 f,f\rangle \ge \delta\langle f,f\rangle$, where $\delta=\frac{a}{L^2}>0$. Hence, T_3 is a positive definite operator in $L_2[0,L]$ for $\sigma\ge 0$. ■

Note that the operator T_3 will be considered on the interval $[0,L]$, which will be divided into two subintervals, with the continuity properties in the sequel. These continuity properties will keep the operator self-adjoint and positive definite.

Example 2.3.2. The difference operator T_6 defined by

$$T_6 f = \left\{ -\frac{1}{h}\left[a(x_{n+1}) \frac{f(x_{n+1}) - f(x_n)}{h} - a(x_n) \frac{f(x_n) - f(x_{n-1})}{h} \right] + \sigma f(x_n) \right\}_{n=1}^{M-1},$$

acting on the grid functions $\{f(x_n)\}_{n=0}^{M}$ defined on the grid space

$$W = \{(x_n) : x_n = nh,\ 0 \le n \le M,\ Mh = L\},$$

satisfying the conditions

$$f(x_1) - f(x_0) = f(x_M) = 0, \tag{2.3}$$

where $a(x_n) \ge a > 0$ for all $0 \le n \le M$ and $\sigma \ge 0$, is self-adjoint and positive definite in $L_2[0,L]_h = L_{2h}[0,L]$.

Solution: Clearly, the operator T_6 is linear. For the sake of brevity, let us introduce the notation $f(x_n) = f_n$. For all $f, g \in D(T_6)$ and $\sigma \ge 0$, we have

$$\begin{aligned} \langle T_6 f, g \rangle &= \sum_{n=1}^{M-1} \left\{ -\frac{1}{h}\left[a_{n+1} \frac{f_{n+1} - f_n}{h} - a_n \frac{f_n - f_{n-1}}{h} \right] + \sigma f_n \right\} g_n h \\ &= -\sum_{n=1}^{M-1} a_{n+1} \frac{f_{n+1} - f_n}{h} g_n + \sum_{n=1}^{M-1} a_n \frac{f_n - f_{n-1}}{h} g_n \\ &\quad + \sigma \sum_{n=1}^{M-1} f_n g_n h. \end{aligned}$$

Applying the Abel's formula and using conditions (2.3), we get

$$\begin{aligned} \langle T_6 f, g \rangle &= -a_M \frac{f_M - f_{M-1}}{h} g_{M-1} - \sum_{n=1}^{M-2} a_{n+1} \frac{f_{n+1} - f_n}{h} g_n \\ &\quad + a_1 \frac{f_1 - f_0}{h} g_0 + \sum_{n=2}^{M-1} a_n \frac{f_n - f_{n-1}}{h} g_n + \sigma \sum_{n=1}^{M-1} f_n g_n h \end{aligned}$$

$$= \frac{a_M}{h} f_{M-1} g_{M-1} + \sum_{n=2}^{M-1} \frac{f_n - f_{n-1}}{h} (a_n g_n - a_n g_{n-1}) + \sigma \sum_{n=1}^{M-1} f_n g_n h. \tag{2.4}$$

Now, denoting $b_n = a_n g_n - a_n g_{n-1}$, we can write

$$\langle T_6 f, g \rangle = \frac{1}{h} \sum_{n=1}^{M} f_n b_n - \frac{1}{h} \sum_{n=1}^{M} f_{n-1} b_n + \sigma \sum_{n=1}^{M-1} f_n g_n h.$$

Applying the Abel's formula once again and using conditions (2.3) it yields

$$\begin{aligned} \langle T_6 f, g \rangle &= \frac{1}{h} \sum_{n=1}^{M} f_n b_n - \frac{1}{h} \sum_{n=1}^{M} f_{n-1} b_n + \sigma \sum_{n=1}^{M-1} f_n g_n h \\ &= \frac{1}{h} \left\{ f_M b_M + \sum_{n=1}^{M-1} f_n b_n - f_0 b_1 - \sum_{n=2}^{M} f_{n-1} b_n \right\} \\ &\quad + \sigma \sum_{n=1}^{M-1} f_n g_n h \\ &= \frac{1}{h} \sum_{n=1}^{M} f_n (b_n - b_{n+1}) + \sigma \sum_{n=1}^{M-1} f_n g_n h. \end{aligned}$$

Finally, denoting back $b_n = a_n g_n - a_n g_{n-1}$, we deduce

$$\begin{aligned} \langle T_6 f, g \rangle &= \sum_{n=1}^{M-1} f_n \left\{ -\frac{1}{h} \left[a_{n+1} \frac{g_{n+1} - g_n}{h} - a_n \frac{g_n - g_{n-1}}{h} \right] + \sigma g_n \right\} h \\ &= \langle f, T_6 g \rangle, \end{aligned}$$

which implies that $T_6^* = T_6$. Hence, T_6 is a self-adjoint operator in $L_{2h}[0, L]$.

Now, we shall show that T_6 is positive definite. The cases $\sigma > 0$ and $\sigma = 0$ will be considered seperately. For all nonzero $f \in D(T_6)$ and $\sigma > 0$, from (2.4) we have

$$\langle T_6 f, f \rangle = \frac{a_M}{h} (f_{M-1})^2 + \sum_{n=1}^{M-1} a_n \left(\frac{f_n - f_{n-1}}{h} \right)^2 h + \sigma \langle f, f \rangle. \tag{2.5}$$

Taking into account that $a(x_n) \geq a > 0$ for all $0 \leq n \leq M$, we obtain

$$\langle T_6 f, f \rangle \geq \sigma \langle f, f \rangle,$$

which means that T_6 is a positive definite operator in $L_{2h}[0,L]$ with $\delta = \sigma > 0$.

Similarly for all nonzero $f \in D(T_6)$ and $\sigma = 0$, taking into account that $a(x_n) \geq a > 0$ for all $0 \leq n \leq M$, from (2.5) we have

$$\langle T_6 f, f\rangle \geq \sum_{n=1}^{M-1} a_n \left(\frac{f_n - f_{n-1}}{h}\right)^2 h \geq a \sum_{n=1}^{M-1} \left(\frac{f_n - f_{n-1}}{h}\right)^2 h.$$

Now, all we need to do is to show that

$$\sum_{n=1}^{M-1} \left(\frac{f_n - f_{n-1}}{h}\right)^2 h \geq \sum_{n=1}^{M-1} (f_n)^2 h = \langle f, f\rangle.$$

To show this, since $f_0 = 0$, we start with

$$\begin{aligned} f_n &= f_n - f_0 = (f_n - f_{n-1}) + (f_{n-1} - f_{n-2}) + \cdots + (f_1 - f_0) \\ &= \sum_{i=1}^{n} (f_i - f_{i-1}) = \sum_{i=1}^{n} \frac{f_i - f_{i-1}}{h} h. \end{aligned}$$

Denoting $m = n - i + 1$, we can write

$$f_n = \sum_{m=1}^{n} \frac{f_{n-m+1} - f_{n-m}}{h} h = \sum_{m=1}^{M-1} \frac{\mu_{n-m+1} - \mu_{n-m}}{h} h,$$

where

$$\mu_{n-m+1} = \begin{cases} f_{n-m+1}, & \text{if } n-m+1 \in \{1, \cdots, M-1\} \text{ or } m \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, to change the order of summations applying the Minkowki's inequality, we have

$$\begin{aligned} \|f\|_{L_{2h}} &= \sqrt{\langle f, f\rangle} = \left\{ \sum_{n=1}^{M-1} (f_n)^2 h \right\}^{1/2} \\ &= \left\{ \sum_{n=1}^{M-1} \left[\sum_{m=1}^{M-1} \frac{\mu_{n-m+1} - \mu_{n-m}}{h} h \right]^2 h \right\}^{1/2} \\ &\leq \sum_{m=1}^{M-1} \left[\sum_{n=1}^{M-1} \left(\frac{\mu_{n-m+1} - \mu_{n-m}}{h} \right)^2 h \right]^{1/2} h. \end{aligned}$$

Now, denoting back $n-m+1=i$, and since $\mu_j = 0$ for all $j \leq 0$ and $j \geq M$, we deduce

$$\begin{aligned}
\sqrt{\langle f,f\rangle} &\leq \sum_{m=1}^{M-1}\left[\sum_{i=2-m}^{M-m}\left(\frac{\mu_i-\mu_{i-1}}{h}\right)^2 h\right]^{1/2} h \\
&= \sum_{m=1}^{M-1}\left[\sum_{i=1}^{M-1}\left(\frac{f_i-f_{i-1}}{h}\right)^2 h\right]^{1/2} h \\
&= \left[\sum_{i=1}^{M-1}\left(\frac{f_i-f_{i-1}}{h}\right)^2 h\right]^{1/2}\sum_{m=1}^{M-1} h \\
&= (M-1)\,h\left[\sum_{i=1}^{M-1}\left(\frac{f_i-f_{i-1}}{h}\right)^2 h\right]^{1/2} \\
&\leq Mh\left[\sum_{i=1}^{M-1}\left(\frac{f_i-f_{i-1}}{h}\right)^2 h\right]^{1/2} \\
&= L\left[\sum_{i=1}^{M-1}\left(\frac{f_i-f_{i-1}}{h}\right)^2 h\right]^{1/2}.
\end{aligned}$$

From that it follows

$$\sum_{n=1}^{M-1}\left(\frac{f_n-f_{n-1}}{h}\right)^2 h \geq \frac{1}{L^2}\sum_{n=1}^{M-1}(f_n)^2 h = \frac{1}{L^2}\langle f,f\rangle.$$

Consequently, for $\sigma = 0$, $\langle T_6 f,f\rangle \geq \delta\langle f,f\rangle$, where $\delta = \frac{a}{L^2} > 0$. Hence, T_6 is a positive definite operator in $L_{2h}[0,L]$ for $\sigma \geq 0$. ■

Note that the operator T_6 will be considered on the interval $[0,L]_h$, which will be divided into two subintervals, with some properties in the sequel. These properties will keep the operator self-adjoint and positive definite.

Definition 2.3.3. Given that H is a Hilbert space over the field of real numbers and that I is any interval of the real line $\mathbb{R}$. A function $f : I \to H$ is called an abstract function. The abstract function $f(t)$ is continuous at the point $t_0 \in I$, if

$$\lim_{t\to t_0}\|f(t)-f(t_0)\|_H = 0.$$

If the abstract function $f(t)$ is continuous throughout the interval I, then it is said to be continuous on I and we use the notation $f \in C[I,H]$.

For example, for the continuous function $f(t,x)$ defined on $[a,b]\times[c,d]$, we can define

the abstract function $f(t,\cdot) = f(t) \in C([a,b], H[c,d])$. Similarly, for the continuous function $f(t,x,y)$ defined on $[a,b] \times [c,d] \times [k,l]$, we can define the abstract function $f(t,\cdot,\cdot) = f(t) \in C([a,b], H([c,d] \times [k,l]))$.

2.4. SPECTRAL REPRESENTATION OF AN OPERATOR

Definition 2.4.1. [36, Definition 3.3-3] A Hilbert space H can be represented by the direct sum of a closed subspace Z and its orthogonal complement $Z^{\perp}$ as

$$H = Z \oplus Z^{\perp}$$

for which

$$H \ni x = y + z, \text{ for some } y \in Z \text{ and } z \in Z^{\perp}.$$

Since the sum is direct, y is unique for any given $x \in H$ and so the spectral representation of unit matrix defines the following linear operator:

$$P : H \to H, \; x \to y = Px.$$

Here, the operator P is called an orthogonal projection or projection on H.

Furthermore, since projections have relatively simple and perspicuous properties, they are used to represent more complicated linear operators on Hilbert spaces.

Theorem 2.4.1 (**Spectral theorem**). [37, p. 21] Let $A : H \to H$ be a self-adjoint linear operator on a complex Hilbert space H. Then, there is a family of an orthogonal projection operators $\{E(\lambda)\}$, $\lambda \in \mathbb{R}$ satisfying the following properties:

(i) $\lambda_1 < \lambda_2$ implies that $E(\lambda_1)E(\lambda_1) = E(\lambda_2)E(\lambda_1) = E(\lambda_1)$

(ii) $E(\lambda + \varepsilon) \overset{\varepsilon \to 0^+}{\longrightarrow} E(\lambda)$ (strongly)

(iii) $E(\lambda) \overset{\lambda \to -\infty}{\longrightarrow} 0$ and $E(\lambda) \overset{\lambda \to +\infty}{\longrightarrow} I$ (strongly).

Then, the operator A has the spectral representation as

$$A = \int_a^b \lambda \, dE_\lambda .$$

Here, E_λ denotes the spectral family generated by the operator A and the integral should be perceived in the sense of uniform operator convergence. By the way, for any $f, g \in H$,

$$\langle Af, g \rangle = \int_a^b \lambda \, dw(\lambda), \; w(\lambda) = \langle E_\lambda f, g \rangle ,$$

where the integral is ordinary Riemann-Stieltjes type.

Corollary 2.4.1. Given that A is a self-adjoint, positive definite (with $\langle Af, f \rangle \geq \delta \langle f, f \rangle$) and linear operator (not necessarily bounded) from its domain $D(A)$, which is dense in a complex Hilbert space H, to H. Then, A and the identity operator have the spectral representations

$$A = \int_\delta^\infty \lambda \, dE_\lambda , \;\; I = \int_\delta^\infty dE_\lambda .$$

More generally, using the spectral representation one may consider a wide class of functions. Namely, if F is a continuously bounded function of such operator, then

$$F(A) = \int_\delta^\infty F(\lambda) \, dE_\lambda . \tag{2.6}$$

Corollary 2.4.2. [37, p. 21] Using representation (3.12), the properties of E_λ and Stieltjes integral, we have the following abstract estimate:

$$\|F(A)\|_{H \to H} \leq \sup_{\delta \leq \lambda < \infty} |F(\lambda)| .$$

Example 2.4.1. Given that A is a positive definite self-adjoint. Then,

$$\begin{aligned} \|\exp(-tA)\|_{H \to H} &\leq 1, \\ \|\exp(-(t-s)A)\|_{H \to H} &\leq 1. \end{aligned}$$

Solution: Since A is a positive definite self-adjoint, using the spectral representation we

can write

$$\exp(-tA) f = \int_{\delta}^{\infty} e^{-\lambda t} \, dE_{\lambda f}.$$

Then, for any $t \geq 0$, we have that

$$\|\exp(-tA)\|_{H \to H} \leq \sup_{\delta \leq \lambda < \infty} \left| e^{-\lambda t} \right| \leq e^{-\delta t} \leq 1.$$

In the same manner, since $s \in [0,t]$,

$$\|\exp(-(t-s)A)\|_{H \to H} \leq \sup_{\delta \leq \lambda < \infty} \left| e^{-\lambda (t-s)} \right| \leq e^{-\delta (t-s)} \leq 1,$$

which completes the solution ■

Example 2.4.2. Let $R = (I + \tau A_h)^{-1}$, where A_h is a positive definite self-adjoint (difference) operator acting on the grid functions in a Hilbert space H_h. Then,

$$\left\| R^k \right\|_{H_h \to H_h} \leq 1$$

for all $k = 1, 2, \cdots$.

Solution: Since A_h is a positive definite self-adjoint operator, we have that

$$\left\| R^k \right\|_{H_h \to H_h} \leq \sup_{\delta \leq \lambda < \infty} \left| (1 + \tau\lambda)^{-k} \right| \leq \frac{1}{(1 + \tau\delta)^k} \leq 1$$

for all $k = 1, 2, \cdots$. ■

2.5. SEMIGROUPS IN BANACH SPACES

Throughout this subsection, the operator A will be considered as an operator whose domain is dense in E and also its inverse A^{-1} exists and is bounded.

Definition 2.5.1. [38, Definition 1.2] A family (operator-function) $U(t)$, $t \geq 0$, of linear bounded operators is called a strongly continuous semigroup, if

(i) $U(t+x) = U(t)U(x) = U(x)U(t)$, for $t, x \geq 0$ with $U(0) = I$

(ii) For each fixed $f \in E$, the function $U(t) f$ is continuous in $t \geq 0$.

Definition 2.5.2. [38, Definition 1.3] The operator $U'(0)$, defined by the formula

$$U'(0) f = \lim_{s \to +0} \frac{(U(s) - I) f}{s},$$

provided that the limit exists for $f \in E$, is said to be the generator of the semigroup $U(t)$.

Remark 2.5.1. In the literature, generally, a semigroup with the generator $-A$ is denoted by $\exp\{-tA\}$, because if $U(t) = \exp\{-tA\}$, then $U'(0) = -A$.

Corollary 2.5.1. [38, p. 6] From the strong continuity of the operator-functions $U(t)$ it follows that its norm is uniformly bounded on any bounded segment $[0, T]$. Moreover, because of the semigroup property, when $t \to \infty$ this norm grows no faster than an exponential. Namely,

$$\|U(t)\|_{E \to E} \leq M e^{\omega t},\ M > 0.$$

Particularly, the norm of the semigroup $\exp\{-tA\}$ decreases exponentially. In other words, it satisfies the decay estimate

$$\|\exp\{-tA\}\|_{E \to E} \leq M e^{-\delta t},\ M, \delta > 0.$$

Definition 2.5.3. [38, Definition 1.4] A strongly continuous semigroup $U(t)$ acting in a complex Banach space is called analytic if it can be continued from the half-line $0 \leq t \leq \infty$ to an operator-function $U(z)$ that is analytic in some sector

$$S_\alpha = \{z : |\arg z| < \alpha,\ 0 < |z| < \infty\},\ \ 0 < \alpha \leq \pi/2,$$

and is strongly continuous in its closure $\overline{S_\alpha}$.

Corollary 2.5.2. The class of analytic semigroups is less wide than the one of strongly continuous semigroups.

Remark 2.5.2. The semigroup $\exp\{-tA\}$ is not only a strongly continuous semigroup but also an analytic semigroup in E.

Theorem 2.5.1. [38, p. 8] The analytic semigroup $\exp\{-tA\}$ is infinitely differentiable in the operator norm for $t > 0$ and its derivatives satisfy the estimate

$$\left\| \frac{d^n \exp\{-tA\}}{dt^n} \right\|_{E\to E} = \|A^n \exp\{-tA\}\|_{E\to E} \leq M^n n^n t^{-n} e^{-\delta t},\ M, \delta, t > 0.$$

Particularly,

$$\left\| \frac{d \exp\{-tA\}}{dt} \right\|_{E\to E} = \|A \exp\{-tA\}\|_{E\to E} \leq M t^{-1} e^{-\delta t} \leq M t^{-1},\ M, \delta, t > 0.$$

2.6. BANACH FIXED-POINT THEOREM

Definition 2.6.1. [40, Definition 1.1] An element x in a set X is called a fixed-point of a mapping $T : X \to X$, if it is mapped onto itself, i.e., $Tx = x$. In other words, it is an element whose image Tx coincides with itself.

Actually, a fixed-point of a mapping T can be obtained by the relation

$$x_{k+1} = Tx_k,\ k = 0, 1, 2, \cdots, \tag{2.7}$$

where x_0 is an arbitrary value. Here, each procedure of this computation is called as iteration.

Definition 2.6.2. [40, Definition 1.2] Assume that $(X, \|\cdot\|)$ is a normed space. A mapping $T : X \to X$ is called a contraction on X if there exists a real number $0 < \alpha < 1$ such that for any $x, y \in X$

$$\|Tx - Ty\| \leq \alpha \|x - y\|.$$

Here, α is called the contractivity factor of T.

Theorem 2.6.1 (**Banach fixed-point theorem**). [36, Theorem 5.1-2] Assume that $(X, \|\cdot\|)$ is a Banach space with $X \neq \varnothing$ and T is a contraction on X. Then, T has precisely one fixed point.

Remark 2.6.1. Obviously, Banach fixed-point theorem can be applied to study the unique solvability of problems involving algebraic equations, ordinary/partial differential

equations, integral equations, *etc*. On the other hand, on the unique solvability of a problem Banach fixed point theorem states a sufficient condition, that is T is a contraction mapping. However, even if the mapping T is not a contraction, it may still have a unique fixed point.

Remark 2.6.2. As it can be checked from (2.7), Banach fixed-point theorem is not just a tool for the investigation of unique solvability but also a constructive process for the unique fixed-point.

Definition 2.6.3. [36, p. 307] A function $f : [a,b] \to [a,b]$ is said to satisfy a Lipschitz condition on $[a,b]$, if there is $c > 0$ such that for all $x, y \in [a,b]$,

$$|f(x) - f(y)| \leq c\,|x - y|\,.$$

Here, the constant c is called as Lipschitz constant.

3. RESULTS

3.1. A TIME-DEPENDENT SOURCE IDENTIFICATION PROBLEM FOR SEMILINEAR PARABOLIC EQUATIONS

In this section, a time-dependent SIP governed by semilinear parabolic equations subject to an integral overdetermination is studied. For solving this problem numerically FDM is applied and the first order of accuracy Rothe difference scheme is considered. Existence and uniqueness results for the solutions of the problem and proposed difference scheme are established using the concepts and tools given in Section 2. The proposed difference scheme is tested on a sample problem for the demonstration of its efficiency. Finally, some numerical discussions and concluding remarks are given.

We shall investigate existence and uniqueness results for the solution $(u(t,x), p(t))$ of the SIP

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right) = p(t)\,q(x) + f(t,x,u), x \in (0,l), \\ \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right) + b(t,x)u(t,x) = p(t)\,q(x) \\ +g(t,x,u),\ x \in (l,L),\ t \in (0,T), \\ u(0,x) = \varphi(x),\ x \in [0,L], \\ u_x(t,0) = u(t,L) = 0,\ \int_0^L u(t,x)\,dx = \psi(t), \\ u(t,l+) = u(t,l-),\ u_x(t,l+) = u_x(t,l-),\ t \in [0,T]. \end{cases} \tag{3.1}$$

Here, $a(x) \geq a > 0$, $f(t,x)$, $\varphi(x)$ and $\psi(t)$ are given sufficiently smooth functions, and $q(x)$ is a sufficiently smooth function assuming $q'(0) = q(L) = 0$ and $\int_0^L q(x)\,dx \neq 0$.

Note that problem (3.1) has been considered for the linear equations, for example, in papers [15–17]. In these papers, several stability inequalities and difference schemes have been presented for the solution of the problem. By the way, the results obtained in this section have been published in [41, 42].

3.1.1. Differential Problem

The unique solvability of problem (3.1) is established by the following theorem. Note that throughout the thesis, K_i's symbolize the positive constants.

Theorem 3.1.1. Assume that there are constants $K_1, K_2 > 0$ such that the continuous functions $f(t,x,u)$ and $g(t,x,u)$ on $[0,T]\times[0,l]$ and $[0,T]\times[l,L]$, respectively, satisfy the Lipschitz conditions

$$\|f(t,\cdot,u(t,\cdot)) - f(t,\cdot,v(t,\cdot))\|_{L_2[0,l]} \leq K_1 \|u(t,\cdot) - v(t,\cdot)\|_{L_2[0,l]}, \tag{3.2}$$

$$\|g(t,\cdot,u(t,\cdot)) - g(t,\cdot,v(t,\cdot))\|_{L_2[l,L]} \leq K_2 \|u(t,\cdot) - v(t,\cdot)\|_{L_2[l,L]} \tag{3.3}$$

for any $t \in [0,T]$ and for any $u, v \in C([0,T], L_2[0,L])$. Then, a unique solution $(u(t,x), p(t))$ of problem (3.1) exists in $C([0,T], L_2[0,L]) \times C[0,T]$, where $C([0,T], L_2[0,L])$ is the space of all continuous functions $\xi(t)$ defined on $[0,T]$ with values in the Hilbert space H, and is endowed with

$$\|\xi\|_{C([0,T],H)} = \max_{0\leq t\leq T} \|\xi(t)\|_H .$$

Proof: In order to solve problem (3.1), the following substitution is employed:

$$u(t,x) = \eta(t) q(x) + w(t,x), \tag{3.4}$$

where

$$\eta(t) = \int_0^t p(y)\,dy, \ \eta(0) = 0, \tag{3.5}$$

and $w(t,x)$ is the solution of the auxilliary problem

$$\begin{cases} \frac{\partial w(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial w(t,x)}{\partial x}\right) = f(t,x,w(t,x)+\eta(t)q(x)) \\ +\frac{\partial}{\partial x}\left(a(x)\eta(t)\frac{\partial q(x)}{\partial x}\right),\ x\in(0,l), \\ \frac{\partial w(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial w(t,x)}{\partial x}\right) + b(t,x)w(t,x) \\ = g(t,x,w(t,x)+\eta(t)q(x)) + \frac{\partial}{\partial x}\left(a(x)\eta(t)\frac{\partial q(x)}{\partial x}\right) \\ -b(t,x)\eta(t)q(x),\ x\in(l,L),\ t\in(0,T), \\ w(0,x)=\varphi(x),\ x\in[0,L], \\ w_x(t,0)=w(t,L)=0, \\ w(t,l+)=w(t,l-),\ w_x(t,l+)=w_x(t,l-),\ t\in[0,T]. \end{cases} \tag{3.6}$$

Note that from the overdetermined condition, substitution (3.4) and equation (3.5) the following formulas for $\eta(t)$ and $p(t)$ are derived:

$$\eta(t) = \frac{\psi(t)-\int_0^L w(t,x)\,dx}{\int_0^L q(x)\,dx}, \tag{3.7}$$

$$p(t) = \eta'(t) = \frac{\psi'(t)-\int_0^L \frac{\partial w(t,x)}{\partial t}dx}{\int_0^L q(x)\,dx}. \tag{3.8}$$

Hence, the proof of Theorem 3.1.1 follows from substitution (3.4), formulas (3.7), (3.8) and the next theorem. ■

Theorem 3.1.2. Suppose that the assumptions of Theorem 3.1.1 are fulfilled, then a unique solution $w(t,x)$ of problem (3.6) exists in $C([0,T],L_2[0,L])$.

Proof: Let us start with introducing the positive definite self-adjoint differential operator A (sf. Example 2.3.1) given by

$$Aw = -\frac{\partial}{\partial x}\left(a(x)\frac{\partial w(x)}{\partial x}\right) \tag{3.9}$$

whose domain is

$$\begin{aligned} D(A) = \{ & w : w, w'' \in L_2[0,L],\ w'(0)=w(L)=0, \\ & w(l-)=w(l+),\ w'(l-)=w'(l+)\}, \end{aligned}$$

and also the bounded operator $B(t)$ defined by the formula

$$B(t)w = \begin{cases} 0, & x \in (0,l), \\ b(t,x)\,w(t,x), & x \in (l,L) \end{cases} \tag{3.10}$$

with the domain $D(B(t)) = C([0,T], L_2[0,L])$. Then, with the help of these operators one can view problem (3.6) in the abstract form

$$\begin{cases} \frac{dw(t)}{dt} + Aw(t) = F(t, w(t) + \eta(t)q) - B(t)w(t) \\ -\eta(t)[A + B(t)]q,\ 0 < t < T,\ w(0) = \varphi, \end{cases} \tag{3.11}$$

where

$$F(t,\cdot, w(t,\cdot) + \eta(t)q(\cdot)) = \left\{ \begin{array}{l} f(t,x,w(t,x) + \eta(t)q(x)),\ 0 < x < l \\ g(t,x,w(t,x) + \eta(t)q(x)),\ l < x < L \end{array} \right\}.$$

Now, for the investigation of unique solvability of problem (3.11) we apply Banach fixed-point theorem which requires us to write the problem in the equivalent operator form

$$w(t) = Fw(t).$$

To do this, applying the Cauchy formula for problem (3.11), we reach

$$\begin{aligned} Fw(t) &= e^{-tA}\varphi + \int_0^t e^{-(t-s)A}\{F(s, w(s) + \eta(s)q) \\ &\quad -B(s)w(s) - \eta(s)[A + B(s)]q\}\,ds \\ &= e^{-tA}\left[\varphi + \int_0^t e^{sA}\{F(s, w(s) + \eta(s)q) \right. \\ &\quad \left. -B(s)w(s) - \eta(s)[A + B(s)]q\}\,ds\right]. \end{aligned} \tag{3.12}$$

Obviously, F from $C([0,T], L_2[0,L])$ into itself is a continuous mapping. Now, we will prove that F is a contraction. In Example 2.1.1, it was shown that the norms

$$\|\varphi\|_{C^*([0,T],H)} = \max_{0 \le t \le T} e^{-\mu t}\,\|\varphi(t)\|_H,$$

and

$$\|\varphi\|_{C([0,T],H)} = \max_{0\le t\le T} \|\varphi(t)\|_H$$

are equivalent.

Applying the triangle inequality, Lipschitz conditions (3.28) and (3.3), formula (3.7) and the trivial estimate (sf. Example 2.4.1)

$$\left\| e^{-(t-s)A} \right\|_{L_2[0,L]\to L_2[0,L]} \le 1,$$

from (3.12) we obtain

$$\begin{aligned}
& e^{-\mu t} \|Fw - Fv\|_{L_2[0,L]} \\
\le & \int_0^t \left\| e^{-(t-s)A} \right\|_{L_2[0,L]\to L_2[0,L]} e^{-\mu(t-s)} e^{-\mu s} \\
& \times \|B(s)[w(s)-v(s)] + F(s, w(s)+\eta(s)q) \\
& - F(s, v(s)+\eta(s)q) + K_3[w(s)-v(s)]\|_{L_2[0,L]} ds \\
\le & K_4 \int_0^t e^{-\mu(t-s)} ds \max_{0\le s\le T} e^{-\mu s} \|w(s)-v(s)\|_{L_2[0,L]} \\
\le & K_4 \frac{1-e^{-\mu T}}{\mu} \|w-v\|_{C^*([0,T],L_2[0,L])}
\end{aligned}$$

for all $t \in [0,T]$ and for all $w, v \in C^*([0,T], L_2[0,L])$. Here,

$$K_4 = \|B(t)\|_{L_2[0,L]\to L_2[0,L]} + \max\{K_1, K_2\} + K_3.$$

Therefore, taking the maximum of the last inequality, we can write

$$\|Fw - Fv\|_{C^*([0,T],L_2[0,L])} \le \alpha_\mu \|w-v\|_{C^*([0,T],L_2[0,L])},$$

where

$$\alpha_\mu = K_4 \frac{1-e^{-\mu T}}{\mu}.$$

Obviously, as $\mu \to \infty$, $\alpha_\mu \to 0$. That means F is a contraction, and so a unique solution $w(t,x)$ of problem (3.6) exists in $C([0,T],L_2[0,L])$ by Banach fixed-point theorem. ■

3.1.2. Difference Problem

In this subsection, for solving problem (3.1) numerically we propose the first order of accuracy Rothe difference scheme. We consider the set of mesh points $S = \{(t_k, x_n)\}$, where

$$t_k = k\tau,\ k = 0,1,\cdots N,\ \tau = \frac{T}{N},$$

and

$$x_n = \begin{cases} nh_0,\ 0 \le n \le M_1,\ M_1 h_0 = l, \\ l + (n - M_1)h,\ M_1 < n \le M,\ (M - M_1)h = L - l. \end{cases}$$

Then, the Rothe difference scheme is given by

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{1}{h_0}\left(a(x_{n+1}) \frac{u_{n+1}^k - u_n^k}{h_0} - a(x_n) \frac{u_n^k - u_{n-1}^k}{h_0} \right) \\ = p_k q(x_n) + f\left(t_k, x_n, u_n^k\right), 1 \le k \le N,\ 1 \le n \le M_1 - 1, \\ \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{1}{h}\left(a(x_{n+1}) \frac{u_{n+1}^k - u_n^k}{h} - a(x_n) \frac{u_n^k - u_{n-1}^k}{h} \right) \\ + b(t_k, x_n) u_n^k = p_k q(x_n) + g\left(t_k, x_n, u_n^k\right), \\ 1 \le k \le N,\ M_1 + 1 \le n \le M, \\ u_n^0 = \varphi(x_n),\ 0 \le n \le M, \\ u_1^k - u_0^k = u_M^k = 0,\ 0 \le k \le N, \\ \sum\limits_{j=1}^{M} u_j^k h^* = \psi(t_k),\ 0 \le k \le N, \\ h_0\left(u_{M_1+1}^k - u_{M_1}^k\right) = h\left(u_{M_1}^k - u_{M_1-1}^k\right),\ 0 \le k \le N. \end{cases} \tag{3.13}$$

Here, we presume that $q_1 - q_0 = q_M = 0$, and $\sum_{j=1}^{M} q_j h^* \neq 0$. Here and throughout this subsection,

$$h^* = \begin{cases} h_0,\ 1 \le j \le M_1, \\ h,\ M_1 + 1 \le j \le M. \end{cases}$$

The unique solvability of difference scheme (3.13) is established by the theorem given below:

Theorem 3.1.3. Assume that f and g satisfy Lipschitz conditions (3.28) and (3.3), respectively. Then, difference scheme (3.13) has a unique solution

$$(U_h^\tau, P^\tau) = \left(\left\{ \left\{ u_n^k \right\}_{n=0}^{M} \right\}_{k=0}^{N}, \left\{ p^k \right\}_{k=1}^{N} \right)$$

in $C_\tau(L_{2h}) \times C[0,T]_\tau$, where $C_\tau(L_{2h}) = C([0,T]_\tau, L_2[0,L]_h)$ is the space of grid functions v_h^τ, endowed with

$$\|v_h^\tau\|_{C_\tau(L_{2h})} = \max_{1 \le k \le N} \left(\sum_{j=0}^{M} \left| v_j^k \right|^2 h^* \right)^{1/2}.$$

Proof: For solving difference scheme (3.13) the following substitution is employed:

$$u_n^k = \eta^k q_n + w_n^k, \tag{3.14}$$

where

$$\eta^k = \sum_{j=1}^{k} p^j \tau,\ 1 \le k \le N,\ \eta^0 = 0,\ q_n = q(x_n), \tag{3.15}$$

and $\left\{w_n^k\right\}$ is the solution of the auxilliary difference scheme

$$\begin{cases} \frac{w_n^k - w_n^{k-1}}{\tau} - \frac{1}{h_0}\left[a\left(x_{n+1}\right)\left(\frac{w_{n+1}^k - w_n^k}{h_0} + \eta^k\left(\frac{q_{n+1}-q_n}{h_0}\right)\right)\right. \\ \left. -a\left(x_n\right)\left(\frac{w_n^k - w_{n-1}^k}{h_0} + \eta^k\left(\frac{q_n - q_{n-1}}{h_0}\right)\right)\right] = f\left(t_k, x_n, w_n^k + \eta^k q_n\right), \\ 1 \le k \le N,\ 1 \le n \le M_1 - 1, \\ \frac{w_n^k - w_n^{k-1}}{\tau} - \frac{1}{h}\left[a\left(x_{n+1}\right)\left(\frac{w_{n+1}^k - w_n^k}{h} + \eta^k\left(\frac{q_{n+1}-q_n}{h}\right)\right)\right. \\ \left. -a\left(x_n\right)\left(\frac{w_n^k - w_{n-1}^k}{h} + \eta^k\left(\frac{q_n - q_{n-1}}{h}\right)\right)\right] + b\left(t_k, x_n\right)\left(w_n^k + \eta^k q_n\right) \\ = g\left(t_k, x_n, w_n^k + \eta^k q_n\right), 1 \le k \le N,\ M_1 + 1 \le n \le M, \\ w_n^0 = \varphi\left(x_n\right),\ 0 \le n \le M, \\ w_1^k - w_0^k = w_M^k = 0,\ 0 \le k \le N, \\ h_0\left(w_{M_1+1}^k - w_{M_1}^k\right) = h\left(w_{M_1}^k - w_{M_1-1}^k\right),\ 0 \le k \le N. \end{cases} \tag{3.16}$$

Note that from the overdetermined condition and substitution (3.14) and equation (3.15) we have the following formulas for η^k and p^k:

$$\eta^k = \frac{\psi\left(t_k\right) - \sum\limits_{j=1}^{M} w_j^k h^*}{\sum\limits_{j=1}^{M} q_j h^*}, \tag{3.17}$$

$$p^k = \frac{\psi\left(t_k\right) - \psi\left(t_{k-1}\right) - \sum\limits_{j=1}^{M}\left(w_j^k - w_j^{k-1}\right) h^*}{\tau \sum\limits_{j=1}^{M} q_j h^*}. \tag{3.18}$$

Thus, the proof of Theorem 3.1.3 follows from substitution (3.14), formulas (3.17), (3.18) and the next theorem. ■

Theorem 3.1.4. Suppose that the assumptions of Theorem 3.1.3 are fulfilled, then a unique solution $W_h^\tau = \left\{\left\{w_n^k\right\}_{n=0}^M\right\}_{k=0}^N$ of difference scheme (3.16) exists in $C_\tau\left(L_{2h}\right)$.

Proof: Firstly, to the differential operators A and $B(t)$ defined by (3.9) and (3.10), we assign the positive definite self-adjoint difference operator A_h (sf. Example 2.3.2) defined

by the formula

$$A_h w_k^h = \left\{ -\frac{1}{h^*} \left[a\left(x_{n+1}\right) \frac{w_{n+1}^k - w_n^k}{h^*} - a\left(x_n\right) \frac{w_n^k - w_{n-1}^k}{h^*} \right] \right\}_{n=1}^{M-1}$$

acting on the grid functions $\left\{w_k^h\right\}_{k=0}^N$ satisfying the conditions

$$w_1^k - w_0^k = w_M^k = 0,\ h_0 \left(w_{M_1+1}^k - w_{M_1}^k \right) = h \left(w_{M_1}^k - w_{M_1-1}^k \right),\ 0 \le k \le N,$$

and the bounded operator $B_h(t_k)$ given by

$$B_h(t_k) w_k^h = \begin{cases} 0, & 1 \le n \le M_1, \\ b\left(t_k, x_n\right) w_n^k, & M_1 + 1 \le n \le M - 1 \end{cases}$$

acting on the grid functions $\left\{w_k^h\right\}_{k=0}^N$. Then, with the help of these operators one can write difference scheme (3.16) in the abstract form

$$\begin{cases} \frac{w_k^h - w_{k-1}^h}{\tau} + A_h w_k^h = F\left(t_k, w_k^h + \eta^k q^h\right) - B_h(t_k) w_k^h \\ -\eta^k \left(A_h + B_h(t_k)\right) q^h,\ 1 \le k \le N\ ,\ w_0^h = \varphi, \end{cases} \tag{3.19}$$

where

$$F\left(t_k, \cdot, w_k^h + \eta^k q^h\right) = \left\{ \begin{array}{c} f\left(t_k, x_n, w_n^k + \eta^k q_n\right), 1 \le n \le M_1 - 1, \\ 0, \\ g\left(t_k, x_n, w_n^k + \eta^k q_n\right), M_1 + 1 \le n \le M \end{array} \right\}.$$

From difference scheme (3.19) it follows that

$$w_k^h = R u_{k-1} + R \left[F\left(t_k, w_k^h + \eta^k q^h\right) - B_h(t_k) w_k^h - \eta^k \left[A_h^x + B_h(t_k)\right] q^h \right] \tau$$

for all $1 \le k \le N$, where $R = \left(I + \tau A_h\right)^{-1}$. Using this recurrence relation one can see that

$$\begin{aligned} w_k^h \ &= \ R^k \varphi + \sum_{i=1}^{k} R^{k-i+1} \left[F\left(t_i, w_i^h + \eta^i q^h\right) \right. \\ &\quad \left. - B_h(t_i) w_i^h - \eta^i \left[A_h^x + B_h(t_i)\right] q^h \right] \tau \end{aligned} \tag{3.20}$$

for all $1 \le k \le N$.

Now, for the investigation of unique solvability of difference scheme (3.19) we apply Banach fixed-point theorem which requires us to write it in the equivalent operator form

$$w_k^h = F^{\tau} w_k^h.$$

To do this, in (3.34) denoting

$$F^{\tau} w_k^h = R^k \varphi + \sum_{i=1}^{k} R^{k-i+1} \left[F\left(t_i, w_i^h + \eta^i q^h\right) - B_h(t_i) w_i^h - \eta^i \left[A_h^x + B_h(t_i)\right] q^h \right] \tau, \quad (3.21)$$

we get

$$w_k^h = F^{\tau} w_k^h.$$

Now, we will prove that F^{τ} is a contraction mapping in $C_{\tau}(L_{2h})$. In Example 2.1.2, it was shown that the norms

$$\|\varphi\|_{C_{\tau}^*(H)} = \max_{1 \le k \le N} \frac{1}{(1+\mu\tau)^k} \|\varphi(t_k)\|_H,$$

and

$$\|\varphi\|_{C_{\tau}(H)} = \max_{1 \le k \le N} \|\varphi(t_k)\|_H$$

are equivalent.

Applying the triangle inequality, Lipschitz conditions (3.28) and (3.3), formula (3.7) and the trivial estimate (sf. Example 2.4.2)

$$\left\| R^k \right\|_{L_{2h}[0,L] \to L_{2h}[0,L]} \le 1,$$

from (3.21) we obtain

$$\begin{aligned} & \frac{1}{(1+\mu\tau)^k} \left\| F^{\tau} w_k^h - F^{\tau} v_k^h \right\|_{L_{2h}[0,L]} \\ \le & \sum_{i=1}^{k} \left\| R^{k-i+1} \right\|_{L_{2h}[0,L] \to L_{2h}[0,L]} \frac{1}{(1+\mu\tau)^k} \frac{(1+\mu\tau)^i}{(1+\mu\tau)^i} \end{aligned}$$

$$
\begin{aligned}
&\times \left[\left\| F\left(t_i, w_i^h + \eta^i q^h\right) - F\left(t_i, v_i^h + \eta^i q^h\right) \right\|_{L_{2h}[0,L]} \right. \\
&+ \|B_h(t_i)\|_{L_{2h}[0,L]\to L_{2h}[0,L]} \left\| w_i^h - v_i^h \right\|_{L_{2h}[0,L]} \\
&\left. + K_5 \left\| w_i^h - v_i^h \right\|_{L_{2h}[0,L]} \right] \tau \\
\leq\ & K_6 \sum_{i=1}^{k} (1+\mu\tau)^{i-k} \tau \max_{1\leq i\leq N} \frac{1}{(1+\mu\tau)^i} \left\| w_i^h - v_i^h \right\|_{L_{2h}[0,L]},
\end{aligned}
$$

where

$$K_6 = \|B_h(t_i)\|_{L_{2h}[0,L]\to L_{2h}[0,L]} + \max\{K_1, K_2\} + K_5.$$

Since, from the geometric series,

$$
\begin{aligned}
\sum_{i=1}^{k} (1+\mu\tau)^{i-k} \tau &= \left[\frac{1}{1-\frac{1}{1+\mu\tau}} - \frac{1}{(1+\mu\tau)^k} \frac{1}{1-\frac{1}{1+\mu\tau}} \right] \tau \\
&= \frac{(1+\mu\tau)^k - 1}{\mu (1+\mu\tau)^{k-1}},
\end{aligned}
$$

taking the maximum of the last inequality, we deduce

$$\|F^{\tau} w - F^{\tau} v\|_{C_\tau^*(L_{2h})} \leq \alpha_\mu \|w - v\|_{C_\tau^*(L_{2h})}$$

for all $w, v \in C_\tau^*(L_{2h})$. Here,

$$\alpha_\mu = K_6 \frac{(1+\mu\tau)^k - 1}{\mu (1+\mu\tau)^{k-1}}.$$

Applying L'Hopital's rule, we deduce

$$\lim_{\mu\to\infty} \alpha_\mu = K_6 \tau,$$

and hence for sufficiently small τ, the contractivity factor $\alpha_\mu < 1$. Then, a unique solution $W_h^\tau = \left\{ \left\{ w_n^k \right\}_{n=0}^{M} \right\}_{k=0}^{N}$ of difference scheme (3.16) exists in $C_\tau^*(L_{2h})$ by Banach fixed-point theorem. ■

Actually, the solution of difference scheme (3.13) is to be obtained from the following

iterative form of formula (3.14):

$$u_n^{k,m} = \eta^{k,m} q_n + w_n^{k,m}, \tag{3.22}$$

where $\left\{ w_n^{k,m} \right\}$ are to be obtained from the following iterative form of the difference scheme (3.16)

$$\begin{cases} \frac{w_n^{k,m} - w_n^{k-1,m}}{\tau} - \frac{1}{h_0} \left[a\left(x_{n+1}\right) \left(\frac{w_{n+1}^{k,m} - w_n^{k,m}}{h_0} + \eta^{k,m} \left(\frac{q_{n+1} - q_n}{h_0} \right) \right) \right. \\ \left. - a\left(x_n\right) \left(\frac{w_n^{k,m} - w_{n-1}^{k,m}}{h_0} + \eta^{k,m} \left(\frac{q_n - q_{n-1}}{h_0} \right) \right) \right] = f\left(t_k, x_n, u_n^{k,m-1}\right), \\ 1 \le k \le N,\ 1 \le n \le M_1 - 1, \\ \frac{w_n^{k,m} - w_n^{k-1,m}}{\tau} - \frac{1}{h} \left[a\left(x_{n+1}\right) \left(\frac{w_{n+1}^{k,m} - w_n^{k,m}}{h} + \eta^{k,m} \left(\frac{q_{n+1} - q_n}{h} \right) \right) \right. \\ \left. - a\left(x_n\right) \left(\frac{w_n^{k,m} - w_{n-1}^{k,m}}{h} + \eta^{k,m} \left(\frac{q_n - q_{n-1}}{h} \right) \right) \right] + b\left(t_k, x_n\right) \left(w_n^{k,m} + \eta^k q_n \right) \\ = g\left(t_k, x_n, u_n^{k,m-1}\right), 1 \le k \le N,\ M_1 + 1 \le n \le M, \\ w_n^{0,m} = \varphi\left(x_n\right),\ 0 \le n \le M, \\ w_1^{k,m} - w_0^{k,m} = w_M^{k,m} = 0,\ 0 \le k \le N, \\ h_0 \left(w_{M_1+1}^{k,m} - w_{M_1}^{k,m} \right) = h \left(w_{M_1}^{k,m} - w_{M_1-1}^{k,m} \right),\ 0 \le k \le N, \end{cases} \tag{3.23}$$

and $\left\{ \eta^{k,m} \right\}$ are to be obtained from the following iterative form of formula (3.17)

$$\eta^{k,m} = \frac{\psi\left(t_k\right) - \sum\limits_{j=1}^{M} w_j^{k,m} h^*}{\sum\limits_{j=1}^{M} q_j h^*}. \tag{3.24}$$

By the way, $\left\{ p^{k,m} \right\}$ are to be obtained from the following iterative form of formula (3.18)

$$p^{k,m} = \frac{\psi\left(t_k\right) - \psi\left(t_{k-1}\right) - \sum\limits_{j=1}^{M} \left(w_j^{k,m} - w_j^{k-1,m} \right) h^*}{\tau \sum\limits_{j=1}^{M} q_j h^*}. \tag{3.25}$$

In these formulas and the difference scheme, $m = 1, 2, \cdots, j$ denotes the number of iteration and the initial guess $u_n^{k,0}, 0 \le k \le N,\ 0 \le n \le M$ is to be made. The iterative computation of this difference scheme is to be terminated when the maximum difference

between the grid points of two successive results becomes less than the prespecified termination criteria ε.

All in all, for solving difference scheme (3.13), we follow the algorithm given below, for $0 \le k \le N$, $0 \le n \le M$:

1) Start with $m = 1$.

2) Using the values $u_n^{k,m-1}$ (initially given for $m = 1$ or calculated in Step 4) the values $w_n^{k,m}$ are calculated.

3) The values $p^{k,m}$ are calculated.

4) The values $u_n^{k,m}$ are calculated.

5) If the maximum difference between the corresponding grid points of $u_n^{k,m-1}$ and $u_n^{k,m}$ is greater than the given tolerance value, take $m = m + 1$ and return Step 2. Otherwise, terminate the iteration process and take the $\left(u_n^{k,m}, p^{k,m}\right)$ as the result of the given problem.

3.2. A SOURCE IDENTIFICATION PROBLEM FOR A SEMILINEAR DIFFERENTIAL EQUATION

In this section, an abstract SIP (a class of SIPs) for a semilinear differential equation with a final overdetermination, which has an application in PDEs, in a Banach space is studied. Moreover, for solving this problem numerically FDM is applied and the first order of accuracy Rothe difference scheme is presented. Results for existence and uniqueness of the solutions of the problem and the proposed difference scheme are established using the concepts and tools given in Section 2. The proposed difference scheme is tested on a sample problem for the demonstration of its efficiency. Finally, some numerical discussions and concluding remarks are given.

We shall investigate results for existence and uniqueness of the solution $(u(t), p)$ of the abstract SIP governed by a semilinear differential equation with a final overdetermination

$$\begin{cases} \frac{du}{dt} + Au(t) = f(t, u(t)) + p,\ t \in (0,T), \\ u(0) = \varphi,\ u(T) = \psi \end{cases} \tag{3.26}$$

in an arbitrary Banach space E. Here, A is a linear operator acting in E whose domain is $D(A)$, with the assumption that $-A$ is the generator of the analytic semigroup $\exp\{-tA\}$ with an exponentially decreasing norm

$$\left\|e^{-tA}\right\|_{E\to E} \leq Me^{-\delta t},\ t\left\|Ae^{-tA}\right\|_{E\to E} \leq M,\ t \geq 0. \tag{3.27}$$

Note that problem (3.26) has been considered for the linear equation, namely when $f = f(t)$, for example in papers [21, 25]. In these papers, several stability inequalities and difference schemes have been presented for the solution of the problem. By the way, the results obtained in this section are published in [43, 44].

3.2.1. Differential Problem

For the theoretical considerations in this subsection, we study on the functional space $C([0,T],E)$, the space of all abstract continuous functions $\xi(t)$ defined on $[0,T]$ with values in a Banach space E, endowed with

$$\|\xi\|_{C([0,T],E)} = \max_{0\leq t\leq T} \|\xi(t)\|_E.$$

Definition 3.2.1. The pair $(u(t),p)$ is called the solution of problem (3.26) in $C([0,T],E)\times E_1$, if

(i) $\frac{du}{dt}, Au(t) \in C([0,T],E)$, $p \in E_1 \subset E$;

(ii) $(u(t),p)$ satisfies the equation and boundary conditions in (3.26).

The unique solvability of problem (3.26) is established by the following theorem:

Theorem 3.2.1. Suppose that $\varphi, \psi \in D(A)$ and the abstract function $f \in C([0,T],E)$ admits the Lipschitz condition

$$\|f(t,u) - f(t,v)\|_E \leq K_7 \|u-v\|_E,\ K_7 > 0 \tag{3.28}$$

for any $t \in [0,T]$ and for any $u, v \in C([0,T],E)$. If the contractivity factor

$$\alpha = \frac{2K_7}{\delta}\max\{1,M\}(1+M) < 1,$$

then problem (3.26) has a unique solution in $C([0,T],E)\times D(A)$.

Proof: For the investigation of unique solvability of problem (3.26) we apply Banach fixed-point theorem which requires us to write the problem in equivalent operator form

$$\begin{pmatrix} u(t) \\ A^{-1}p \end{pmatrix} = B \begin{pmatrix} u(t) \\ A^{-1}p \end{pmatrix}.$$

To do this, applying the Cauchy formula for problem (3.26), we get

$$\begin{aligned} u(t) &= e^{-tA}\varphi + \int_0^t e^{-(t-s)A}\left[f(s,u(s))+p\right]ds \\ &= e^{-tA}\varphi + \int_0^t e^{-(t-s)A} f(s,u(s))\,ds + A^{-1}\left(I-e^{-tA}\right)p \qquad (3.29) \\ &= e^{-tA}\left[\varphi + \int_0^t e^{sA} f(s,u(s))\,ds\right] + A^{-1}\left(I-e^{-tA}\right)p. \end{aligned}$$

Since the semigroup $\exp\{-tA\}$ obeys exponential decay estimate (3.27), the inverse of the operator $I-e^{-TA}$ exists and is bounded [36, Theorem 7.3-1]. Hence, applying the condition $u(T)=\psi$, we get

$$u(T) = e^{-TA}\varphi + \int_0^t e^{-(T-s)A} f(s,u(s))\,ds + A^{-1}\left(I-e^{-TA}\right)p = \psi,$$

or

$$A^{-1}p = \left(I-e^{-TA}\right)^{-1}\left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s,u(s))\,ds\right). \qquad (3.30)$$

Then, applying formulas (3.29) and (3.30), we can write

$$\begin{aligned} u(t) &= e^{-tA}\varphi + \int_0^t e^{-(t-s)A} f(s,u(s))\,ds + \left(I-e^{-tA}\right)\left(I-e^{-TA}\right)^{-1} \\ &\quad \times \left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s,u(s))\,ds\right) \qquad (3.31) \end{aligned}$$

for all $0 \leq t \leq T$.

Now, let us introduce the continuous mapping B from $C([0,T],E) \times E$ into itself such that

$$\begin{pmatrix} u(t) \\ A^{-1}p \end{pmatrix} = B \begin{pmatrix} u(t) \\ A^{-1}p \end{pmatrix}$$

where

$$B \begin{pmatrix} u(t) \\ \\ A^{-1}p \end{pmatrix} = \begin{pmatrix} e^{-tA}\varphi + \int_0^t e^{-(t-s)A} f(s,u(s))\,ds \\ +\beta\left(I-e^{-tA}\right)\left(I-e^{-TA}\right)^{-1}A^{-1}p \\ +(1-\beta)\left(I-e^{-tA}\right)\left(I-e^{-TA}\right)^{-1} \\ \times\left[\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s,u(s))\,ds\right] \\ \\ \left(I-e^{-TA}\right)^{-1}\left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s,u(s))\,ds\right) \end{pmatrix}.$$

Here $\beta \in [0,1]$. Putting $\beta = 0$, and denoting

$$\begin{aligned} B_1u(t) &= e^{-tA}\varphi + \int_0^t e^{-(t-s)A} f(s,u(s))\,ds + \left(I-e^{-tA}\right)\left(I-e^{-TA}\right)^{-1} \\ &\quad \times \left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s,u(s))\,ds\right), \\ B_2u(t) &= \left(I-e^{-TA}\right)^{-1}\left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s,u(s))\,ds\right), \end{aligned}$$

we get

$$B \begin{pmatrix} u(t) \\ \\ A^{-1}p \end{pmatrix} = \begin{pmatrix} B_1u(t) \\ \\ B_2u(t) \end{pmatrix}.$$

Applying estimate (3.27) and the formula

$$\left(I-e^{-TA}\right)^{-1}=I-e^{-TA}+e^{-2TA}-\cdots,$$

we get

$$\begin{aligned}\left\|\left(I-e^{-TA}\right)^{-1}\right\|_{E\to E} &\leq 1+Me^{-\delta T}+Me^{-2\delta T}+\cdots \\ &\leq \max\{1,M\}\frac{1}{1-e^{-\delta T}}. \end{aligned} \tag{3.32}$$

Thus, applying the triangle inequality, estimates (3.27) and (3.32), we get

$$\begin{aligned}&\|B_1u(t)-B_1v(t)\|_E \\ \leq\ & \left\|\left(I-e^{-TA}\right)^{-1}\right\|_{E\to E}\left\{\int_0^t\left\|e^{-(t-s)A}-e^{-(T-s)A}\right\|_{E\to E}\|f(s,u(s))-f(s,v(s))\|_E\,ds\right. \\ &\left.+\int_t^T\left\|\left(I-e^{-tA}\right)e^{-(T-s)A}\right\|_{E\to E}\|f(s,u(s))-f(s,v(s))\|_E\,ds\right\} \\ \leq\ & \max\{1,M\}\frac{K_7}{1-e^{-\delta T}}(1+M)\left\{\int_0^t e^{-\delta(t-s)}\|u(s)-v(s)\|_E\,ds\right. \\ &\left.+\int_t^T e^{-\delta(T-s)}\|(u(s)-v(s))\|_E\,ds\right\} \\ \leq\ & \frac{2K_7}{\delta}\max\{1,M\}(1+M)\max_{0\leq s\leq T}\|(u(s)-v(s))\|_E \\ =\ & \frac{2K_7}{\delta}\max\{1,M\}(1+M)\|u-v\|_{C([0,T],E)}, \end{aligned}$$

$$\begin{aligned}&\|B_2u(t)-B_2v(t)\|_E \\ \leq\ & \left\|\left(I-e^{-TA}\right)^{-1}\right\|_{E\to E}\int_0^T\left\|e^{-(T-s)A}\right\|_{E\to E}\|f(s,u(s))-f(s,v(s))\|_E\,ds \\ \leq\ & \max\{1,M\}\frac{K_7}{1-e^{-\delta T}}\int_0^T e^{-\delta(T-s)}\|u(s)-v(s)\|_E\,ds \\ \leq\ & \frac{K_7}{\delta}\max\{1,M\}\max_{0\leq s\leq T}\|(u(s)-v(s))\|_E \\ =\ & \frac{K_7}{\delta}\max\{1,M\}\|u-v\|_{C([0,T],E)} \end{aligned}$$

for any $t \in [0,T]$. Hence, we get

$$\begin{aligned} & \max\left\{\left\|B_1u(t)-B_1v(t)\right\|_E, \left\|B_2u(t)-B_2v(t)\right\|_E\right\} \\ \leq\ & \frac{2K_7}{\delta}\max\{1,M\}(1+M)\left\|u-v\right\|_{C([0,T],E)}. \end{aligned}$$

Therefore, if the contractivity factor

$$\alpha = \frac{2K_7}{\delta}\max\{1,M\}(1+M) < 1,$$

then problem (3.26) has a unique solution in $C([0,T],E)\times D(A)$ by Banach fixed-point theorem. ■

Moreover, the pair $\left(u(t),A^{-1}p\right)$ can be obtained by

$$\left(u(t),A^{-1}p\right) = \lim_{m\to\infty}\left(u_m(t),A^{-1}p_m\right),$$

where $\left(u_m(t),A^{-1}p_m\right)$, $m=1,2,\cdots$ are defined by

$$\begin{aligned} u_m(t) &= e^{-tA}\varphi + \int_0^t e^{-(t-s)A}f(s,u_{m-1}(s))\,ds + \left(I-e^{-tA}\right)\left(I-e^{-TA}\right)^{-1} \\ &\quad \times\left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A}f(s,u_{m-1}(s))\,ds\right), \\ A^{-1}p_m &= \left(I-e^{-TA}\right)^{-1}\left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A}f(s,u_{m-1}(s))\,ds\right), \end{aligned}$$

and $u_0(t)\in C([0,T],D(A))$ is given.

3.2.2. Difference Problem

For the theoretical considerations in this subsection, we study on the functional space $C([0,T]_\tau,E)$, the space of grid functions $\xi^\tau = \{\xi_k\}_0^N$ defined on $[0,T]_\tau$ with values in a Banach space E and is endowed with

$$\left\|\xi^\tau\right\|_{C([0,T]_\tau,E)} = \max_{0\leq k\leq N}\left\|\xi_k\right\|_E.$$

For solving problem (3.26) numerically, the first order of accuracy Rothe difference scheme

$$\begin{cases} \frac{u_k-u_{k-1}}{\tau}+Au_k=f(t_k,u_k)+p,\ 1\le k\le N, \\ t_k=k\tau,\ N\tau=T,\ u_0=\varphi,\ u_N=\psi \end{cases} \tag{3.33}$$

is considered.

The unique solvability of problem (3.33) is established by the following theorem:

Theorem 3.2.2. Let the conditions of Theorem 3.2.1 be fulfilled, then a unique solution of difference scheme (3.33) exists in $C([0,T]_\tau,E)$.

Proof: For the investigation of unique solvability of problem (3.33) we apply Banach fixed-point theorem which requires us to write the problem in equivalent operator form

$$u_k=B^\tau u_k.$$

From difference scheme (3.33) it follows that

$$u_k=Ru_{k-1}+R\left[f(t_k,u_k)+p\right]\tau$$

for all $1\le k\le N$, where $R=(I+\tau A)^{-1}$. Using this recurrence relation one can see that

$$u_k=R^k\varphi+\sum_{i=1}^{k}R^{k-i+1}f(t_i,u_i)\tau+A^{-1}\left(I-R^k\right)p \tag{3.34}$$

for all $1\le k\le N$. Applying the condition $u_N=\psi$, we obtain

$$u_N=R^N\varphi+\sum_{i=1}^{N}R^{N-i+1}f(t_i,u_i)\tau+A^{-1}\left(I-R^N\right)p=\psi,$$

or

$$A^{-1}p=\left(I-R^N\right)^{-1}\left\{\psi-R^N\varphi-\sum_{i=1}^{N}R^{N-i+1}f(t_i,u_i)\tau\right\}. \tag{3.35}$$

Then, substituting (3.35) in (3.34), we can write that

$$\begin{aligned} u_k &= R^k\varphi + \sum_{i=1}^{k} R^{k-i+1} f(t_i,u_i)\tau + \left(I-R^k\right)\left(I-R^N\right)^{-1} \\ &\quad \times \left\{ \psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f(t_i,u_i)\tau \right\}. \end{aligned}$$

Now, let us introduce the mapping B^τ that maps $C\left([0,T]_\tau, E\right)\times E$ onto itself, defined by

$$B^\tau \begin{pmatrix} u_k \\ \\ A^{-1}p \end{pmatrix} = \begin{pmatrix} R^k\varphi + \sum_{i=1}^{k} R^{k-i+1} f(t_i,u_i)\tau + \left(I-R^k\right)\left(I-R^N\right)^{-1} \\ \times \left\{ \psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f(t_i,u_i)\tau \right\} \\ \\ \left(I-R^N\right)^{-1}\left\{ \psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f\left(t_i,u_i\right)\tau \right\} \end{pmatrix}.$$

Denoting

$$\begin{aligned} B_1^\tau u_k &= R^k\varphi + \sum_{i=1}^{k} R^{k-i+1} f(t_i,u_i)\tau + \left(I-R^k\right)\left(I-R^N\right)^{-1} \\ &\quad \times \left\{ \psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f(t_i,u_i)\tau \right\}, \\ B_2^\tau u_k &= \left(I-R^N\right)^{-1}\left\{ \psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f\left(t_i,u_i\right)\tau \right\}, \end{aligned}$$

we get

$$B^\tau \begin{pmatrix} u_k \\ \\ A^{-1}p \end{pmatrix} = \begin{pmatrix} B_1^\tau u_k \\ \\ B_2^\tau u_k \end{pmatrix}.$$

Applying estimate [38, p. 99]

$$\left\| R^k \right\|_{E\to E} \le \frac{M}{\left(1+\delta\tau\right)^k},\ k \ge 1, \tag{3.36}$$

and the formula

$$\left(I-R^N\right)^{-1} = I + R^N + R^{2N} + \cdots,$$

we get

$$\begin{aligned}\left\|\left(I-R^N\right)^{-1}\right\|_{E\to E} &\leq 1+\frac{M}{(1+\delta\tau)^N}+\frac{M}{(1+\delta\tau)^{2N}}+\cdots \\ &\leq \frac{\max\{1,M\}}{1-\frac{1}{(1+\delta\tau)^N}}. \end{aligned} \tag{3.37}$$

Applying the triangle inequality, Lipschitz condition (3.28) and estimates (3.36), (3.37), we reach

$$\begin{aligned}&\|B_1^\tau u_k - B_1^\tau v_k\|_E \\ \leq\ & \left\|\left(I-R^N\right)^{-1}\right\|_{E\to E}\left\{\sum_{i=1}^{k}\left\|R^{k-i+1}-R^{N-i+1}\right\|_{E\to E}\|f(t_i,u_i)-f(t_i,v_i)\|_E\,\tau\right. \\ &\left.+\sum_{i=k+1}^{N}\left\|\left(I-R^k\right)R^{N-i+1}\right\|_{E\to E}\|f(t_i,u_i)-f(t_i,v_i)\|_E\,\tau\right\} \\ \leq\ & \max\{1,M\}\frac{K_7}{1-\frac{1}{(1+\delta\tau)^N}}(1+M) \\ &\times\left\{\sum_{i=1}^{k}\frac{1}{(1+\delta\tau)^{k-i+1}}\|f(t_i,u_i)-f(t_i,v_i)\|_E\,\tau\right. \\ &\left.+\sum_{i=k+1}^{N}\frac{1}{(1+\delta\tau)^{N-i+1}}\|f(t_i,u_i)-f(t_i,v_i)\|_E\,\tau\right\} \\ \leq\ & \frac{2K_7}{\delta}\max\{1,M\}(1+M)\max_{1\leq i\leq N}\|u_i-v_i\|_E \\ =\ & \frac{2K_7}{\delta}\max\{1,M\}(1+M)\|u^\tau-v^\tau\|_{C([0,T]_\tau,E)}, \end{aligned}$$

$$\begin{aligned}&\|B_2^\tau u_k - B_2^\tau v_k\|_E \\ \leq\ & \left\|\left(I-R^N\right)^{-1}\right\|_{E\to E}\sum_{i=1}^{N}\left\|R^{N-i+1}\right\|_{E\to E}\|f(t_i,u_i)-f(t_i,v_i)\|_E\,\tau \\ \leq\ & \max\{1,M\}\frac{K_7}{1-\frac{1}{(1+\delta\tau)^N}}\sum_{i=1}^{N}\frac{1}{(1+\delta\tau)^{N-i+1}}\|f(t_i,u_i)-f(t_i,v_i)\|_E\,\tau \\ \leq\ & \frac{K_7}{\delta}\max\{1,M\}\max_{1\leq i\leq N}\|u_i-v_i\|_E \\ =\ & \frac{K_7}{\delta}\max\{1,M\}\|u^\tau-v^\tau\|_{C([0,T]_\tau,E)} \end{aligned}$$

for any $k = 1,2,...,N$. Hence, we get

$$\begin{aligned} & \max\left\{\left\|B_1^\tau u_k - B_1^\tau v_k\right\|_E, \left\|B_2^\tau u_k - B_2^\tau v_k\right\|_E\right\} \\ \leq & \frac{2K_7}{\delta}\max\left\{1,M\right\}(1+M)\left\|u^\tau - v^\tau\right\|_{C([0,T]_\tau,E)}. \end{aligned}$$

Thus, if the contractivity factor

$$\alpha = \frac{2K_7}{\delta}\max\left\{1,M\right\}(1+M) < 1,$$

then a unique solution of problem (3.33) exists in $C([0,T]_\tau,E)$ by Banach fixed-point theorem. ■

Moreover, the pair $\left(u^\tau, A^{-1}p\right)$ can be obtained by

$$\left(u^\tau, A^{-1}p\right) = \lim_{m\to\infty}\left(u^{\tau,m}, A^{-1}p^m\right),$$

where $\left(u^{\tau,m}, A^{-1}p^m\right)$, $m = 1,2,\cdots$ are defined by the formulas

$$\begin{aligned} u^{\tau,m} &= R^k\varphi + \sum_{i=1}^{k} R^{k-i+1} f(t_i, u_i^{m-1})\tau + \left(I - R^k\right)\left(I - R^N\right)^{-1} \\ &\quad \times \left(\psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f(t_i, u_i^{m-1})\tau\right), \\ A^{-1}p_m &= \left(I - R^N\right)^{-1}\left(\psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f\left(t_i, u_i^{m-1}\right)\tau\right), \end{aligned}$$

where $u_i^0 \in C([0,T]_\tau, D(A))$ is given.

3.2.3. Application

We shall study a particular case of abstract Theorems 3.2.1 and 3.2.2. Let us consider the space-dependent SIP for a semilinear parabolic equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \sigma u(t,x) = p(x) \\ + f(t,x,u(t,x)),\ 0 < x < 1,\ 0 < t < T, \\ u(0,x) = \varphi(x),\ u(T,x) = \psi(x),\ 0 \leq x \leq 1, \\ u(t,0) = u(t,1) = 0,\ 0 \leq t \leq T, \end{cases} \tag{3.38}$$

where $u(t,x)$, and $p(x)$ are the unknown vector functions, $\varphi(x)$, $\psi(x)$ and $a(x) \geq a > 0$ are given sufficiently smooth functions and $\sigma \geq 0$.

Clearly, for this application the operator A is

$$A = -a(x)\frac{d^2}{dx^2} + \sigma I, \tag{3.39}$$

acting in $C[0,1]$. This operator admits [38, 39] the requirements given in the description of problem 3.26.

Theorem 3.2.3. Suppose that $\varphi(x), \psi(x)$ are two elements in $C_2[0,1]$ and the continuous function $f(t,x,u)$ on $[0,T] \times [0,1]$ admits the Lipschitz condition

$$\|f(t,\cdot,u(t,\cdot)) - f(t,\cdot,v(t,\cdot))\|_{C[0,1]} \leq K_7 \|u(t,\cdot) - v(t,\cdot)\|_{C[0,1]},\ K_7 > 0 \tag{3.40}$$

for any $t \in [0,T]$ and for any $u, v \in C([0,T], C[0,1])$. If the contractivity factor

$$\alpha = \frac{2K_7}{\delta}\max\{1,M\}(1+M) < 1,$$

then a unique solution of problem (3.38) exists in $C([0,T], C[0,1]) \times C_2[0,1]$.

For solving problem (3.38) numerically, we start with defining the set of grid points

$$\begin{aligned} S &= \{(t_k,x_n) : t_k = k\tau,\ 0 \leq k \leq N,\ N\tau = T, \\ &\qquad x_n = nh,\ 0 \leq n \leq M,\ Mh = 1\}. \end{aligned}$$

Then, in the light of abstract Rothe difference scheme (3.33) with the difference analogue of operator (3.39)

$$Au = \left\{ -a(x_n)\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + \sigma u_n \right\}_{n=1}^{M-1}, \tag{3.41}$$

the following Rothe difference scheme is considered:

$$\begin{cases} \dfrac{u_n^k - u_n^{k-1}}{\tau} - a(x_n)\dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \sigma u_n^k = \phi_n^k, \\ \phi_n^k = p_n + f(t_k, x_n, u_n^k),\ 1 \le k \le N,\ 1 \le n \le M-1, \\ u_n^0 = \varphi(x_n),\ u_n^N = \psi(x_n),\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N. \end{cases} \tag{3.42}$$

Theorem 3.2.4. Suppose that the assumptions of Theorem 3.2.3 are fulfilled, then difference scheme (3.42) has a unique solution.

Actually, the solution of difference scheme (3.42) is to be obtained from the following iterative form of the difference scheme:

$$\begin{cases} \dfrac{u_n^{k,m} - u_n^{k-1,m}}{\tau} - a(x_n)\dfrac{u_{n+1}^{k,m} - 2u_n^{k,m} + u_{n-1}^{k,m}}{h^2} + \sigma u_n^{k,m} = \phi_n^{k,m-1}, \\ \phi_n^{k,m-1} = p_n^m + f(t_k, x_n, u_n^{k,m-1}),\ 1 \le k \le N,\ 1 \le n \le M-1, \\ u_n^{0,m} = \varphi(x_n),\ u_n^{N,m} = \psi(x_n),\ 0 \le n \le M, \\ u_0^{k,m} = u_M^{k,m} = 0,\ 0 \le k \le N. \end{cases} \tag{3.43}$$

Here, $m = 1, 2, \cdots, j$ denotes the number of iterations and the initial guess $u_n^{k,0}, 0 \le k \le N,\ 0 \le n \le M$ is to be made. The iterative computation of this difference scheme is to be terminated when the maximum difference between the grid points of two successive results becomes less than the prespecified termination criteria ε.

Theorem 3.2.5. [25, Theorem 2] In implementation of difference scheme (3.43) the formula

$$p^{h,m} = A\psi^h - Av_N^{h,m},$$

where A is the operator given in (3.41), is to be applied.

More precisely,

$$\begin{aligned} p_n^m &= -a(x_n)\frac{\psi_{n+1} - 2\psi_n + \psi_{n-1}}{h^2} + \sigma\psi_n \\ &\quad + a(x_n)\frac{v_{n+1}^{N,m} - 2v_n^{N,m} + v_{n-1}^{N,m}}{h^2} - \sigma v_n^{N,m}, \end{aligned} \tag{3.44}$$

for $1 \leq n \leq M-1$. Here, values of $v_{n+1}^{N,m}$, $v_n^{N,m}$, $v_{n-1}^{N,m}$ are obtained from the following iterative difference scheme:

$$\begin{cases} \dfrac{v_n^{k,m} - v_n^{k-1,m}}{\tau} - a(x_n) \dfrac{v_{n+1}^{k,m} - 2v_n^{k,m} + v_{n-1}^{k,m}}{h^2} + \sigma v_n^{k,m} = \Psi_n^{k,m-1}, \\ \Psi_n^{k,m-1} = f(t_k, x_n, u_n^{k,m-1}),\ 1 \leq k \leq N,\ 1 \leq n \leq M-1, \\ v_n^{0,m} - v_n^{N,m} = \varphi(x_n) - \psi(x_n),\ 0 \leq n \leq M, \\ v_0^{k,m} = v_M^{k,m} = 0,\ 0 \leq k \leq N. \end{cases} \tag{3.45}$$

For solving difference scheme (3.42), we follow the numerical steps of article [25]. Actually, for $0 \leq k \leq N$, $0 \leq n \leq M$, the algorithm is as follows:

1) Start with $m = 1$.

2) Using the values $u_n^{k,m-1}$ (initially given for $m = 1$ or calculated in Step 4) the values $v_n^{k,m}$ are calculated.

3) The values p_n^m are calculated.

4) The values $u_n^{k,m}$ are calculated.

5) If the maximum difference between the corresponding grid points of $u_n^{k,m-1}$ and $u_n^{k,m}$ is greater than the given tolerance value, take $m = m+1$ and return Step 2. Otherwise, terminate the iteration process and take the $\left(u_n^{k,m}, p_n^m\right)$ as the result of the given problem.

4. DISCUSSION

4.1. NUMERICAL DISCUSSION FOR PROBLEM (3.1)

In the previous section, the Rothe difference scheme was proposed for solving problem (3.1) numerically. To support the theoretical considerations and for the discussion of the numerical results, we create the following test problem:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(e^{-x}\frac{\partial u(t,x)}{\partial x}\right) = p(t)\cos\left(\frac{\pi}{2}x\right) + f(t,x,u), \\ f(t,x,u) = e^{-t}\left(x^2-x^3\right) + e^{-t-x}\left(3x^2-8x+2\right) - e^{t}\cos\left(\frac{\pi}{2}x\right) \\ -\frac{1}{2}\sin\left(e^{-t}\left(x^3-x^2\right)\right) + \frac{1}{2}\sin\left(u(t,x)\right),\ t\in(0,1],\ x\in(0,\pi/4), \\ \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(e^{-x}\frac{\partial u(t,x)}{\partial x}\right) + e^{t-x}u(t,x) = p(t)\cos\left(\frac{\pi}{2}x\right) \\ +g(t,x,u),\ g(t,x,u) = e^{-t}\left(x^2-x^3\right) + e^{-t-x}\left(3x^2-8x+2\right) \\ -e^{t}\cos\left(\frac{\pi}{2}x\right) + e^{-t}\left(x^2-x^3\right) + e^{-t-x}\left(3x^2-8x+2\right) - e^{t}\cos\left(\frac{\pi}{2}x\right) \\ +e^{-x}\left(x^3-x^2\right) - \frac{1}{2}\sin\left(e^{-t}\left(x^3-x^2\right)\right) + \frac{1}{2}\sin\left(u(t,x)\right), \\ t\in(0,1],\ x\in(\pi/4,1), \\ u(0,x) = x^3-x^2,\ x\in[0,1], \\ u_x(t,0) = u(t,1) = 0,\ t\in[0,1], \\ u(t,\frac{\pi}{4}+) = u(t,\frac{\pi}{4}-),\ u_x(t,\frac{\pi}{4}+) = u_x(t,\frac{\pi}{4}-),\ t\in[0,1], \\ \int_0^1 u(t,x)dx = \frac{-e^{-t}}{12},\ t\in[0,1]. \end{cases} \tag{4.1}$$

Clearly, $(u,p) = \left(e^{-t}(x^3-x^2), e^{t}\right)$ is the solution of this problem.

For test problem (4.1) we apply the algorithm given in the related subsection. On the set of mesh points $S = \{(t_k, x_n)\}$, where

$$t_k = k\tau,\ 0\le k\le N,\ N\tau = 1,$$

and

$$x_n = \begin{cases} nh_0,\ 0\le n\le M_1,\ M_1h_0 = \pi/4, \\ \pi/4 + (n-M_1)h,\ M_1 < n\le M,\ (M-M_1)h = 1-\pi/4, \end{cases}$$

applying iterative difference scheme (3.23), we consider

$$\begin{cases} \frac{w_n^{k,m}-w_n^{k-1,m}}{\tau}-\frac{1}{h_0}\left\{e^{-x_{n+1}}\left[\frac{w_{n+1}^{k,m}-w_n^{k,m}}{h_0}+\eta^{k,m}\left(\frac{\cos\left(\frac{\pi}{2}x_{n+1}\right)-\cos\left(\frac{\pi}{2}x_n\right)}{h_0}\right)\right]\right. \\ \left.-e^{-x_n}\left[\frac{w_n^{k,m}-w_{n-1}^{k,m}}{h_0}+\eta^{k,m}\left(\frac{\cos\left(\frac{\pi}{2}x_n\right)-\cos\left(\frac{\pi}{2}x_{n-1}\right)}{h_0}\right)\right]\right\}=f\left(t_k,x_n,u_n^{k,m-1}\right), \\ 1\le k\le N,\ 1\le n\le M_1-1, \\ \frac{w_n^{k,m}-w_n^{k-1,m}}{\tau}-\frac{1}{h}\left\{e^{-x_{n+1}}\left[\frac{w_{n+1}^{k,m}-w_n^{k,m}}{h}+\eta^{k,m}\left(\frac{\cos\left(\frac{\pi}{2}x_{n+1}\right)-\cos\left(\frac{\pi}{2}x_n\right)}{h}\right)\right]\right. \\ \left.-e^{-x_n}\left[\frac{w_n^{k,m}-w_{n-1}^{k,m}}{h}+\eta^{k,m}\left(\frac{\cos\left(\frac{\pi}{2}x_n\right)-\cos\left(\frac{\pi}{2}x_{n-1}\right)}{h}\right)\right]\right\} \\ +e^{t_k-x_n}\left[w_n^{k,m}+\eta^{k,m}\cos\left(\frac{\pi}{2}x_n\right)\right]=g\left(t_k,x_n,u_n^{k,m-1}\right), \\ 1\le k\le N,\ M_1+1\le n\le M, \\ w_n^{0,m}=(x_n)^3-(x_n)^2,\ 0\le n\le M, \\ w_1^{k,m}-w_0^{k,m}=w_M^{k,m}=0,\ 0\le k\le N, \\ h_0\left(w_{M_1+1}^{k,m}-w_{M_1}^{k,m}\right)=h\left(w_{M_1}^{k,m}-w_{M_1-1}^{k,m}\right),\ 0\le k\le N,\ m=1,2,\cdots, \end{cases}$$

for which we take the grid function $u_n^{k,0}=\overrightarrow{0}$ and the termination criteria $\varepsilon=10^{-8}$. This iterative difference scheme can be viewed as a system of linear equations with $(M+1)\times(N+1)$ unknowns, whose matrix representation is

$$\begin{cases} A_kW^{k,m}+BW^{k-1,m}=G^{k,m-1}, 1\le k\le N, \\ W^{0,m}=\left\{(x_n)^3-(x_n)^2\right\}_{n=0}^{M},\ m=1,2,\cdots. \end{cases} \tag{4.2}$$

Here A_k and B are $(M+1)\times(M+1)$ square matrices, $W^{k,m}$ and $G^{k,m-1}$ are $(M+1)\times 1$ vectors. Note that the subscript k in A_k indicates that as k changes, so does the coefficient matrix A_k. Let us illustrate these matrices:

$$A_k = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ b_1 & \widetilde{c}_1 & \widetilde{d}_1 & \widetilde{e}_1 & \cdots & \widetilde{e}_1 & \widetilde{e}_1 & \widetilde{e}_1 & \cdots & \widetilde{e}_1 \\ 0 & \widetilde{b}_2 & \widetilde{c}_2 & \widetilde{d}_2 & \cdots & \widetilde{e}_2 & \widetilde{e}_2 & \widetilde{e}_2 & \cdots & \widetilde{e}_2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \widetilde{e}_{M_1-1} & \cdots & \widetilde{b}_{M_1-1} & \widetilde{c}_{M_1-1} & \widetilde{d}_{M_1-1} & \widetilde{e}_{M_1-1} & \widetilde{e}_{M_1-1} & \cdots & \widetilde{e}_{M_1-1} \\ 0 & 0 & \cdots & 0 & h & -(h+h_0) & h_0 & 0 & \cdots & 0 \\ 0 & \widehat{e}_{M_1+1} & \cdots & \widehat{e}_{M_1+1} & \widehat{e}_{M_1+1} & \widehat{b}_{M_1+1} & \widehat{c}^k_{M_1+1} & \widehat{d}_{M_1+1} & \cdots & \widehat{e}_{M_1+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \widehat{e}_{M-1} & \cdots & \widehat{e}_{M-1} & \widehat{e}_{M-1} & \widehat{e}_{M-1} & \widehat{e}_{M-1} & \widehat{b}_{M-1} & \widehat{c}^k_{M-1} & \widehat{d}_{M_1-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1/\tau & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1/\tau & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1/\tau & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$G^{k,m-1} = \begin{bmatrix} 0 \\ f_1^{k,m-1} \\ f_2^{k,m-1} \\ \vdots \\ f_{M_1-1}^{k,m-1} \\ 0 \\ g_{M_1+1}^{k,m-1} \\ g_{M_1+2}^{k,m-1} \\ \vdots \\ g_{M-1}^{k,m-1} \\ 0 \end{bmatrix},$$

where

$$
\begin{aligned}
b_1 &= \frac{e^{-x_1}}{h_0^2},\\
\widetilde{e}_n &= \frac{1}{\sum_{j=1}^{M}\cos\left(\frac{\pi}{2}x_j\right)}\left\{e^{-x_{n+1}}\left[\frac{\cos\left(\frac{\pi}{2}x_{n+1}\right)-\cos\left(\frac{\pi}{2}x_n\right)}{h_0^2}\right]\right.\\
&\quad \left.-e^{-x_n}\left[\frac{\cos\left(\frac{\pi}{2}x_n\right)-\cos\left(\frac{\pi}{2}x_{n-1}\right)}{h_0^2}\right]\right\},\\
\widetilde{b}_n &= \frac{-e^{-x_n}}{h_0^2}+\widetilde{e}_n,\ \widetilde{c}_n=\frac{1}{\tau}+\frac{e^{-x_{n+1}}+e^{-x_n}}{h_0^2}+\widetilde{e}_n,\\
\widetilde{d}_n &= \frac{-e^{-x_{n+1}}}{h_0^2}+\widetilde{e}_n,\\
\widehat{e}_n &= \frac{1}{\sum_{j=1}^{M}q\left(x_j\right)}\left\{e^{-x_{n+1}}\left[\frac{\cos\left(\frac{\pi}{2}x_{n+1}\right)-\cos\left(\frac{\pi}{2}x_n\right)}{h^2}\right]\right.\\
&\quad \left.-e^{-x_n}\left[\frac{\cos\left(\frac{\pi}{2}x_n\right)-\cos\left(\frac{\pi}{2}x_{n-1}\right)}{h^2}\right]\right\},\\
\widehat{b}_n &= \frac{-e^{-x_n}}{h^2}+\widehat{e}_n,\\
\widehat{c}_n^k &= \frac{1}{\tau}+\frac{e^{-x_{n+1}}+e^{-x_n}}{h^2}+e^{-t_k-x_n}+\widehat{e}_n,\\
\widehat{d}_n &= \frac{-e^{-x_{n+1}}}{h^2}+\widehat{e}_n.
\end{aligned}
$$

From (4.2) it follows that

$$W^{k,m}=(A_k)^{-1}\left(G^{k,m-1}-BW^{k-1,m}\right),\ k=1,\cdots,N,\ m=1,2,\cdots. \tag{4.3}$$

Hence, in each iteration after obtaining the values of $W^{k,m}$, using formulas (3.43) and (3.25) the values of $u_n^{k,m}$ and $p^{k,m}$ are obtained.

Now, we present the errors between exact solution and approximate solution, derived by the proposed difference schemes, of problem (4.1). In Tables 3.1 and 3.2, that are constructed for different values of T and $N=M=\frac{4M_1}{3}$, the errors are computed by the formula

$$E_u=\max_{1\le k\le N}\left(\sum_{n=0}^{M}\left|u_n^{k,j}-u(t_k,x_n)\right|^2h^*\right)^{1/2},$$

$$E_p = \max_{1\leq k\leq N} \left| p^{k,j} - p(t_k) \right|,$$

assuming that the iterations are terminated at the $j-$th iteration. Here, $u_n^{k,j}$ and $p^{k,j}$ stand for the approximate solutions and $u(t_k, x_n)$ and $p(t_k)$ stand for the exact solutions, respectively.

Table 4.1: Error analysis for $T = 1$.

	$N = M = 20$	$N = M = 40$	$N = M = 80$
Number of iterations (m)	7	7	7
E_p	0.2566	0.1200	0.0456
E_u	3.1190×10^{-4}	1.0353×10^{-4}	2.9012×10^{-5}

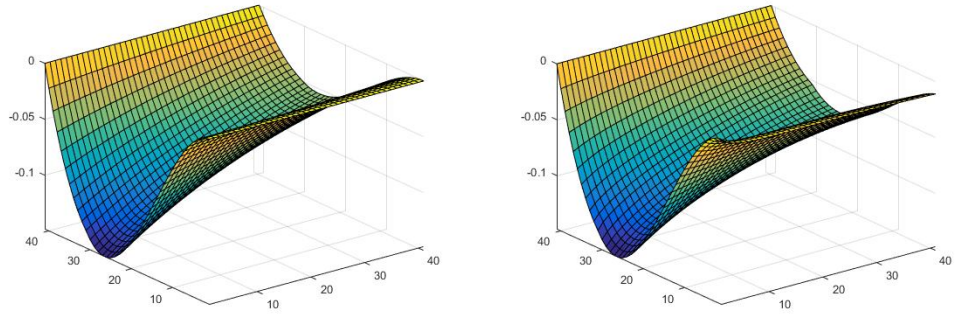
Table 4.2: Error analysis for $T = 2$.

	$N = M = 20$	$N = M = 40$	$N = M = 80$
Number of iterations (m)	7	7	7
E_p	0.9955	0.5008	0.2283
E_u	0.0040	0.0014	4.4448×10^{-4}

These numerical experiments support the theoretical considerations and in the light of numerical experiments and the values in the tables we can conclude that

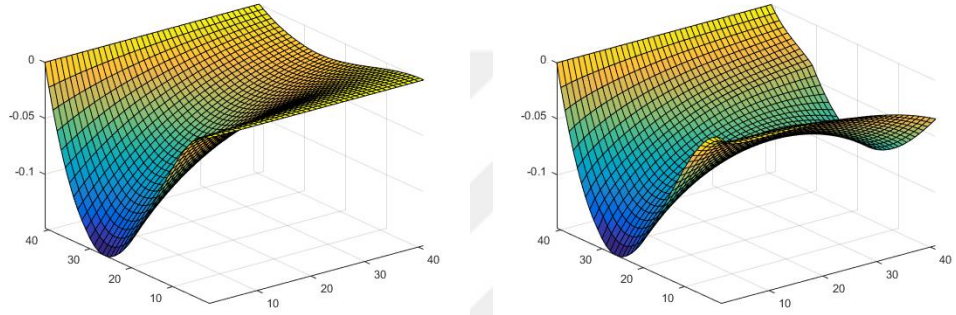
- when N and M are increased, more accurate results are obtained but the number of iterations remains almost same;
- particularly, when N and M are doubled, the errors are decreased by the factor $1/2$ approximately;
- for the greater values of T, namely in the long time run, there is no significant change in the number of iterations;
- convergence of the solutions is validated in the long time run.

The MATLAB implementations used for these computations are given in Appendix 1.



(a) Exact solution (b) Approximate solution

Figure 4.1: Exact and approximate solutions when $T = 1$ and $N = M = 40$.



(a) Exact solution (b) Approximate solution

Figure 4.2: Exact and approximate solutions when $T = 2$ and $N = M = 40$.

4.2. NUMERICAL DISCUSSION FOR PROBLEM (3.38)

In the previous section, the Rothe difference scheme was proposed for solving problem (3.38) numerically. To support the theoretical considerations and for the discussion of the numerical results, let us consider the space-dependent SIP governed by a semilinear parabolic equation subject to a final overdetermination

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - e^{-x}\frac{\partial^2 u(t,x)}{\partial x^2} + 2u(t,x) = p(x) + f(t,x,u), \\ f(t,x,u) = \left(x^2 - x - 2e^{-x}\right)e^{-t} - \cos(\pi x) \\ + \frac{1}{2}\left[\sin\left(e^t u(t,x)\right) - \sin\left(x^2 - x\right)\right],\ x \in (0,1),\ t \in (0,T], \\ u(0,x) = x^2 - x,\ u(T,x) = e^{-T}\left(x^2 - x\right),\ x \in [0,1], \\ u(t,0) = u(t,1) = 0,\ t \in [0,T]. \end{cases} \tag{4.4}$$

Clearly, $(u,p) = \left(e^{-t}\left(x^2 - x\right), \cos(\pi x)\right)$ is the solution of this problem.

For test problem (4.4) we apply the algorithm given in the related subsection. Applying the proposed iterative difference scheme (3.45), we have

$$\begin{cases} \dfrac{v_n^{k,m}-v_n^{k-1,m}}{\tau}-e^{-x_n}\dfrac{v_{n+1}^{k,m}-2v_n^{k,m}+v_{n-1}^{k,m}}{h^2}+2v_n^{k,m}=\Psi_n^{k,m-1}, \\ \Psi_n^{k,m-1}=f(t_k,x_n,u_n^{k,m-1}),\ 1\le k\le N,\ 1\le n\le M-1, \\ v_n^{N,m}-v_n^{0,m}=\left(e^{-T}-1\right)\left(x_n^2-x_n\right),\ 0\le n\le M, \\ v_0^{k,m}=v_M^{k,m}=0,\ 0\le k\le N. \end{cases} \tag{4.5}$$

The matrix representation of this difference scheme is

$$A_n\,V_{n+1}^m+B_n\,V_n^m+C_n\,V_{n-1}^m=F_n^{m-1},\ 1\le n\le M-1,\ V_0^m=V_M^m=\overrightarrow{0}. \tag{4.6}$$

Note that the subscript n in A_n, B_n, C_n indicates that as n changes, so does these matrices. Here, F_n^{m-1} is an $(N+1)\times 1$ column vector, A_n, B_n, C_n are square matrices of the size $(N+1)\times(N+1)$, given below:

$$A_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix} = C_n,$$

$$B_n = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 1 \\ d & b_n & 0 & 0 & \cdots & 0 \\ 0 & d & b_n & 0 & \cdots & 0 \\ 0 & 0 & d & b_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d & b_n \end{bmatrix},$$

$$F_n^{m-1} = \begin{bmatrix} \left(e^{-T}-1\right)\left(x_n^2 - x_n\right) \\ \Psi_n^{1,m-1} \\ \Psi_n^{2,m-1} \\ \vdots \\ \Psi_n^{N,m-1} \end{bmatrix},$$

where

$$a_n = \frac{e^{-x_n}}{h^2},\ b_n = \frac{1}{\tau} + \frac{2e^{-x_n}}{h^2} + 2,\ d = -\frac{1}{\tau},\ 1 \le n \le M.$$

For solving matrix equation (4.6) we apply the modified Gauss elimination method [45], in which the solution is searched in the form

$$V_n^m = \alpha_{n+1}^m V_{n+1}^m + \beta_{n+1}^m,\ n = M-1, \cdots 1,\ V_0^m = V_M^m = \overrightarrow{0},$$

for $m = 1, 2, \cdots j$, where α_n^m 's and β_n^m 's matrices of the size $(N+1)\times(N+1)$ and vectors of the size $(N+1)\times 1$, respectively. In order to find α_n^m 's and β_n^m 's we employ the formulas

$$\begin{aligned} \alpha_{n+1}^m &= -\left(B_n + C_n\alpha_n^m\right)^{-1} A_n, \\ \beta_{n+1}^m &= -\left(B_n + C_n\alpha_n^m\right)^{-1}\left(F_n^{m-1} - C_n\beta_n^{m-1}\right), \end{aligned}$$

where $n = 1, \cdots, M-1$, α_1^m and β_1^{m-1} are the zero matrices and zero vectors.

Then, the values of p_n^m are to be determined by

$$p_n^m = 2e^{-T}\left(x_n^2 - x_n - e^{-x_n}\right) + e^{-x_n}\frac{v_{n+1}^{N,m} - 2v_n^{N,m} + v_{n-1}^{N,m}}{h^2} - 2v_n^{N,m}, \tag{4.7}$$

for $1 \le n \le M-1$, that is obtained from formula (3.44).

Finally, we apply the proposed iterative difference scheme (3.43)

$$\begin{cases} \dfrac{u_n^{k,m}-u_n^{k-1,m}}{\tau}-e^{-x_n}\dfrac{u_{n+1}^{k,m}-2u_n^{k,m}+u_{n-1}^{k,m}}{h^2}+2u_n^{k,m}=\phi_n^{k,m-1}, \\ \phi_n^{k,m-1}=f(t_k,x_n,u_n^{k,m-1})+p_n^m,\ 1\le k\le N,\ 1\le n\le M-1, \\ u_n^{0,m}=x^2-x,\ u_n^{N,m}=e^{-T}\left(x_n^2-x_n\right),\ 0\le n\le M, \\ u_0^{k,m}=u_M^{k,m}=0,\ 0\le k\le N. \end{cases} \tag{4.8}$$

If the values of p_n^m are substituted into difference scheme (4.8), it becomes direct. Then, we write it in the matrix form as

$$A_n\,U_{n+1}^m+D_n\,U_n^m+C_n\,U_{n-1}^m=G_n^{m-1},\ 1\le n\le M-1,\ U_0^m=U_M^m=\overrightarrow{0}, \tag{4.9}$$

where A_n, C_n were the matrices defined above and

$$D_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ d & b_n & 0 & 0 & \cdots & 0 \\ 0 & d & b_n & 0 & \cdots & 0 \\ 0 & 0 & d & b_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d & b_n \end{bmatrix},$$

$$G_n^{m-1} = \begin{bmatrix} x_n^2-x_n \\ \phi_n^{1,m-1} \\ \phi_n^{2,m-1} \\ \vdots \\ \phi_n^{N,m-1} \end{bmatrix}.$$

Matrix equation (4.9) is solved by the modified Gauss elimination method described above. In computations the initial guess is chosen as identical zero grid function $u_n^{k,0}=\overrightarrow{0}$ and when the maximum difference between the grid points of two successive results of iterative difference scheme (4.8) becomes less than the termination criteria $\varepsilon=10^{-8}$, the iterative process is terminated.

In Tables 4.1-4.3, the errors are computed by the formulas

$$E_{u_j} = \max_{\substack{1\le k\le N \\ 1\le n\le M-1}} \left| u_n^{k,j} - u(t_k, x_n) \right|,$$

$$E_{p_j} = \max_{1\le n\le M-1} \left| p_n^j - p(x_n) \right|,$$

assuming that the iterations are terminated at the $j-th$ iteration. Tables are constructed for different T and $N = M$ values.

Table 4.3: Error analysis for $T = 1$.

	$N = M = 20$	$N = M = 40$	$N = M = 80$
Number of iterations (m)	7	7	6
E_p	0.1744	0.0871	0.0435
E_u	0.0052	0.0025	0.0012

Table 4.4: Error analysis for $T = 2$.

	$N = M = 20$	$N = M = 40$	$N = M = 80$
Number of iterations (m)	10	9	9
E_p	0.1249	0.0632	0.0317
E_u	0.0073	0.0038	0.0019

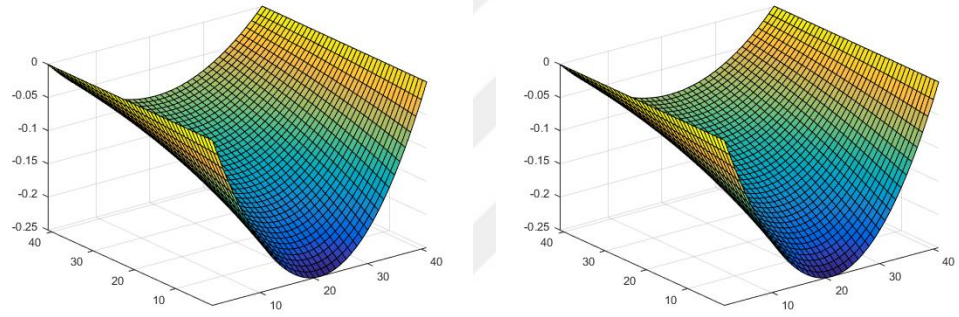
Table 4.5: Error analysis for $T = 3$.

	$N = M = 20$	$N = M = 40$	$N = M = 80$
Number of iterations (m)	22	21	20
E_p	0.1008	0.0529	0.0270
E_u	0.0075	0.0040	0.0021

These numerical experiments support the theoretical considerations and in the light of numerical experiments and the values in the tables we can conclude that

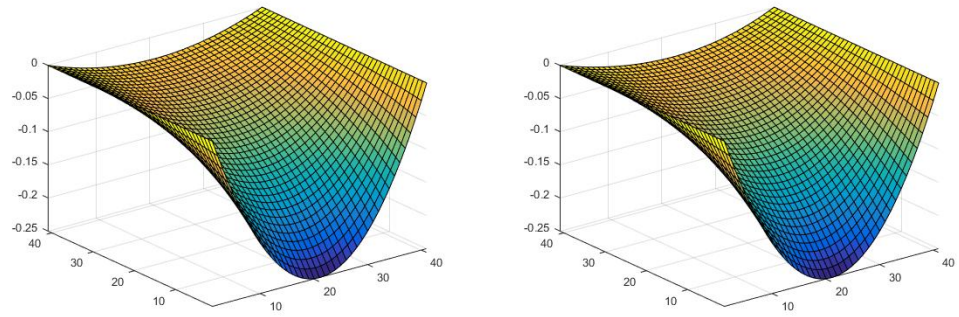
- when N and M are increased, more accurate results are obtained but the number of iterations remains almost same;
- particularly, when N and M are doubled, the errors are decreased by the factor 1/2 approximately;
- for the greater values of T, namely in the long time run, the number of iterations are increased exponentially but there is no significant change in errors;
- convergence of the solutions is validated in the long time run, as well.

The MATLAB implementations used for these computations are given in Appendix 2.



(a) Exact solution (b) Approximate solution

Figure 4.3: Exact and approximate solutions when $T = 1$ and $N = M = 40$.



(a) Exact solution (b) Approximate solution

Figure 4.4: Exact and approximate solutions when $T = 2$ and $N = M = 40$.

5. CONCLUSION AND RECOMMENDATIONS

In this thesis, it is aimed to make some noteworthy contributions to the literature by studying two complementary SIPs: a time-dependent SIP for semilinear parabolic equations subject to an integral overdetermination and an abstract SIP (a class of SIPs) for a semilinear differential equation with a final overdetermination, which has an application in PDEs.

Particularly, in this thesis,

- the results for existence and uniqueness of the solutions of these problems are obtained under the Lipschitz condition;
- the Rothe difference schemes are presented for the numerical solutions of these problems;
- the results for existence and uniqueness of the solutions of difference schemes are obtained under the Lipschitz condition;
- some numerical experiments are conducted for showing the efficiency of the proposed difference schemes;
- the numerical results validate the theoretical considerations of the thesis.

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APPENDICES

APPENDIX 1. Matlab implementation for problem (4.1)

```
function ptsemilinear(N,M1,M)

close; close;

if nargin<1, N=40; M=40; M1=30; end

h0=pi/(4*M1); h=(1-pi/4)/(M-M1); tau=1/N;

A=zeros(M+1,M+1,N+1);

sum=0;

for i=1:M1+1,

x=i*h0;

sum=sum+q(x)*h0; end

for i=M1+2:M+1,

x=pi/4+(i-M1-1)*h;

sum=sum+q(x)*h; end

for i=2:N+1,

t=(i-1)*tau;

for n=2:M1,

x=(n-1)*h0;

A(n,n-1,i)=-a(x)/h0^2;

A(n,n,i)=1/tau+(a(x+h0)+a(x))/h0^2;

A(n,n+1,i)=-a(x+h0)/h0^2;
```

```
for j=2:M+1,
A(n,j,i)=A(n,j,i)+h0*(a(x+h0)*(q(x+h0)-q(x))/h0^2
...-a(x)*(q(x)-q(x-h0))/h0^2)/sum;
end end
for n=M1+2:M,
x=pi/4+(n-M1-1)*h;
A(n,n-1,i)=-a(x)/h^2;
A(n,n,i)=1/tau+(a(x+h)+a(x))/h^2+b(t,x);
A(n,n+1,i)=-a(x+h)/h^2;
for j=2:M+1,
A(n,j,i)=A(n,j,i)+h*(a(x+h)*(q(x+h)-q(x))/h^2
...-a(x)*(q(x)-q(x-h))/h^2-b(t,x)*q(x))/sum;
end end
A(1,1,i)=-1; A(1,2,i)=1; A(M+1,M+1,i)=1;
A(M1+1,M1,i)=h; A(M1+1,M1+1,i)=-h-h0;
A(M1+1,M1+2,i)=h0;
end
B=zeros(M+1,M+1);
for n=2:M1,
B(n,n)=-1/tau; end
for n=M1+2:M,
```

```
B(n,n)=-1/tau; end

U(:,:,1)=zeros(M+1,N+1);

m=2; difference=1;

while difference>=10^-8,

F(:,:,m-1)=zeros(M+1,N+1);

for k=2:N+1,

t=(k-1)*tau;

for n=2:M1,

x=(n-1)*h0;

F(n,k,m-1)=f(t,x)+0.5*sin(U(n,k,m-1))

...+Psy(t)*(a(x+h0)*(q(x+h0)-q(x))/h0^2

...-a(x)*(q(x)-q(x-h0))/h0^2)/sum;

end

for n=M1+2:M,

x=pi/4+(n-M1-1)*h;

F(n,k,m-1)=g(t,x)+0.5*sin(U(n,k,m-1))

...+Psy(t)*(a(x+h)*(q(x+h)-q(x))/h^2

...-a(x)*(q(x)-q(x-h))/h^2-b(t,x)*q(x))/sum;

end end

for n=1:M1+1,

x=(n-1)*h0;
```

```
W(n,1,m)=(x^3-x^2); end
for n=M1+2:M+1,
x=pi/4+(n-M1-1)*h;
W(n,1,m)=(x^3-x^2); end
for k=2:N+1,
W(:,k,m)=inv(A(:,:,k))*(F(:,k,m-1)
...-B*W(:,k-1,m)); end
for k=1:N+1,
t=(k-1)*tau;summ=0;
for n=2:M1+1,
summ=summ+W(n,k,m)*h0; end
for n=M1+2:M+1,
summ=summ+W(n,k,m)*h; end
for n=1:M1+1,
x=(n-1)*h0;
U(n,k,m)=W(n,k,m)+(Psy(t)-summ)*q(x)/sum;
end
for n=M1+2:M+1,
x=pi/4+(n-M1-1)*h;
U(n,k,m)=W(n,k,m)+(Psy(t)-summ)*q(x)/sum;
end end
```

```
difference=max(max(abs(U(:,:,m)-U(:,:,m-1))));
m=m+1; end
pe(1)=1;
for k=2:N+1,
t=(k-1)*tau;sumw=0;
for n=2:M1+1,
sumw=sumw+(W(n,k,m-1)-W(n,k-1,m-1))*h0;
end
for n=M1+2:M+1,
sumw=sumw+(W(n,k,m-1)-W(n,k-1,m-1))*h;
end
pe(k)=(Psy(t)-Psy(t-tau)-sumw)/(tau*sum);
end
for k=1:N+1,
t=(k-1)*tau;
for n=1:M1+1,
x=(n-1)*h0;
es(n,k)=exact(t,x); end
for n=M1+2:M+1,
x=pi/4+(n-M1-1)*h;
es(n,k)=exact(t,x); end end
```

```
for k=1:N+1,
t=(k-1)*tau; pexact(k)=p(t); end
% maxerror=max(max(abs(es-U(:,:,m-1))))
for k=1:N+1,
erroru(k)=0;
for n=1:M1+1,
erroru(k)=erroru(k)+(abs(U(n,k,m-1)
...-es(n,k)).^2)*h0;
end
for n=M1+2:M+1,
erroru(k)=erroru(k)+(abs(U(n,k,m-1)
...-es(n,k)).^2)*h;
end end
maxerroru=max(erroru.1/2)
perror=max(abs(pexact-pe))
m-1
figure ; surf(U(:,:,m-1)); rotate3d ;
axis tight;
figure; surf(es); rotate3d ;
axis tight;
function ax=a(x)
```

```
ax=exp(-x); end

function btx=b(t,x)

btx=exp(t-x); end

function pt=p(t)

pt=exp(t); end

function qx=q(x)

qx=cos(pi*x/2); end

function psyt=Psy(t)

psyt=-exp(-t)/12; end

function ftx=f(t,x)

ftx=exp(-t)*(x^2-x^3)-exp(t)*cos(pi*x/2)

...+exp(-t-x)*(3*x^2-8*x+2)

...-0.5*sin(exp(-t)*(x^3-x^2));

end

function gtx=g(t,x)

gtx=exp(-t)*(x^2-x^3)-exp(t)*cos(pi*x/2)+exp(-t-x)

...*(3*x^2-8*x+2)+exp(-x)*(x^3-x^2)

...-0.5*sin(exp(-t)*(x^3-x^2));

end

function estx=exact(t,x)

estx=exp(-t)*(x^3-x^2);end end
```

APPENDIX 2. Matlab implementation for problem (4.4)

```
function pxsemilinear(N,M)

close; close;

if nargin<2; N=40; M=40; end;

T=1; h=1/M; tau=T/N;

u(:,:,1)=zeros(N+1, M+1);

m=0; difference=1; tolerance=10^-8;

x=0:h:1; a=exp(-x); sg=2;

while difference>tolerance,

m=m+1;

%finding auxilary V

clear A B C V fii p

A=zeros(N+1,N+1,M+1) ;

for j=1:M;

for n=2:N+1

A(n,n,j)=-a(j+1)/h^2; end; end;

C=A;

B=zeros(N+1,N+1,M+1) ;

for j=1:M;

for n=2:N+1;

B(n,n,j)=1/tau+2*a(j+1)/h^2+sg;
```

```
B(n,n-1,j)=-1/tau; end;

B(1,1,j)=-1;B(1,N+1,j)=1; end;

fii=zeros(N+1,M+1);

for n=2:M

x=(n-1)*h;

for k=2:N+1;

t=(k-1)*tau;

fii(k,n)=(x^2-x-2*exp(-x))*exp(-t)-cos(pi*x)

...+sin(exp(t)*u(k,n,m))/2-sin(x^2-x)/2; end;

fii(1,n)=(exp(-T)-1)*(x^2-x); end;

D=eye(N+1);

alpha{2}= zeros(N+1,N+1) ;

beta{2}=zeros(N+1,1) ;

for n=3:M+1;

alpha{n}=-inv(B(:,:,n-1)+C(:,:,n-1)*alpha{n-1})

...*A(:,:,n-1);

beta{n}=inv(B(:,:,n-1)+C(:,:,n-1)*alpha{n-1})

...*(D*(fii(:,n-1))-C(:,:,n-1)*beta{n-1}); end;

for k=1:N+1;

V(k,1)=0;

V(k,M+1)=0; end;
```

```
for n=M:-1:2;

V(:,n)=alpha{n+1}*V(:,n+1)+beta{n+1}; end;

%FINDING P

p(1)=1;

p(M+1)=-1;

for n=2:M;

x=(n-1)*h;

p(n)=2*exp(-T)*(x^2-x-exp(-x))+a(n-1)

...*(V(N+1,n+1)-2*V(N+1,n)+V(N+1,n-1))/h^2

...-sg*V(N+1,n); end;

%FINDING U

clear A; clear B; clear C; clear alpha;

clear betha; clear gamma;

A=zeros(N+1,N+1,M+1) ;

for j=1:M;

for n=2:N+1;

A(n,n,j)=-a(j+1)/h^2; end; end;

C=A;

B=zeros(N+1,N+1,M+1) ;

for j=1:M;

for n=2:N+1;
```

```
B(n,n,j)=1/tau+2*a(j+1)/h^2+sg;

B(n,n-1,j)=-1/tau; end;

B(1,1,j)=1; end;

fii2=zeros(N+1,M+1);

for n=1:M+1

for k=2:N+1;

fii2(k,n)=fii(k,n)+p(n); end;

x=(n-1)*h;

fii2(1,n)=x^2-x; end;

D=eye(N+1);

alpha{2}= zeros(N+1,N+1) ;

beta{2}=zeros(N+1,1);

for n=3:M+1;

alpha{n}=-inv(B(:,:,n-1)+C(:,:,n-1))

...*alpha{n-1}*A(:,:,n-1);

beta{n}=inv(B(:,:,n-1)+C(:,:,n-1)*alpha{n-1})

...*(D*(fii2(:,n-1))-C(:,:,n-1)*beta{n-1}); end;

for k=1:N+1;

u(k,1,m+1)=0;

u(k,M+1,m+1)=0; end;

for n=M:-1:2;
```

```
u(:,n,m+1)=alpha{n+1}*u(:,n+1,m+1)+beta{n+1}; end;

difference=max(max(abs(u(:,:,m+1)-u(:,:,m)))); end

%'EXACT SOLUTION OF PDE' ;

for j=1:M+1;

for k=1:N+1;

t=(k-1)*tau;

x=(j-1)*h;

es(k,j)= exact(t,x); end; end;

%'EXACT SOLUTION OF P' ;

for j=1:M+1;

x=(j-1)*h;

esp(j)=exactp(x); end;

iteration=m

maxerrorp=max(max(abs(esp-p)))

maxerror=max(max(abs(es-u(:,:,m+1))))

figure ; surf(es) ; rotate3d ; axis tight;

title 'EXACT SOLUTION';

figure ; surf(u(:,:,m+1)) ; rotate3d ; axis tight;

title 'DIFFERENCE SCHEMES SOLUTION';

%FUNCTIONS ;

function estx=exact(t,x)
```

```
estx=exp(-t)*(x^2-x);

function espx=exactp(x)

espx=cos(pi*x);
```

CURRICULUM VITAE



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