Universal Journal of Mathematics and Applications, 1 (1) (2018) 1-9

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma

The Application Domain of Infinite Matrices with Algorithms

Murat Kirişci[a*]

[a]*Department of Mathematical Education, Hasan Ali Yücel Education Faculty, Istanbul University, Vefa, 34470, Fatih, Istanbul, Turkey*
[*]*Corresponding author E-mail: mkirisci@hotmail.com, murat.kirisci@istanbul.edu.tr*

Article Info

Keywords: *Algorithm, Flowchart, Matrix domain, Dual space, Schauder basis, AK-property, Monotone norm*
2010 AMS: *Primary 47A15, Secondary 46A45, 46B45, 68Q25*
Received: *1 January 2018*
Accepted: *17 January 2018*
Available online: *24 January 2018*

Abstract

The purpose of this paper is twofold. First, we define the new spaces and investigate some topological and structural properties. Also, we compute dual spaces of new spaces which are help us in the characterization of matrix mappings. Second, we give some examples related to new spaces. A flow chart of the stages of the newly constructed sequence spaces and the algorithms of the workings at each step are given.

1. Introduction

It is well known that, the ω denotes the family of all real (or complex)-valued sequences. ω is a linear space and each linear subspace of ω (with the included addition and scalar multiplication) is called a *sequence space* such as the spaces c, c_0 and ℓ_∞, where c, c_0 and ℓ_∞ denote the set of all convergent sequences in fields $\mathbb{R}$ or $\mathbb{C}$, the set of all null sequences and the set of all bounded sequences, respectively. It is clear that the sets c, c_0 and ℓ_∞ are the subspaces of the ω. Thus, c, c_0 and ℓ_∞ equipped with a vector space structure, from a sequence space. By bs and cs, we define the spaces of all bounded and convergent series, respectively.

A coordinate space (or $K-space$) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space X with a linear topology is called a K-space provided each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A $K-$space is called an $FK-space$ provided X is a complete linear metric space. An $FK-space$ whose topology is normable is called a $BK-$ *space*.

Let X be a $BK-$space. Then X is said to have monotone norm if $\|x^{[m]}\| \geq \|x^{[n]}\|$ for $m > n$ and $\|x\| = \sup \|x^{[m]}\|$. The spaces c_0, c, ℓ_∞, cs, bs have monotone norms.

If a normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + ... + \alpha_n b_n)\| = 0$$

then (b_n) is called *Schauder basis* for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$. An $FK-space$ X is said to have AK property, if $\phi \subset X$ and $\{e^k\}$ is a basis for X, where e^k is a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$ and $\phi = span\{e^k\}$, the set of all finitely non-zero sequences. An $FK-$space $X \supset \phi$ is said to have AB, if $(x^{[n]})$ is a bounded set in X for each $x \in X$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} and $x = (x_k) \in \omega$, where $k, n \in \mathbb{N}$. Then the sequence Ax is called as the $A-$transform of x defined by the usual matrix product. Hence, we transform the sequence x into the sequence $Ax = \{(Ax)_n\}$ where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1.1}$$

***Email addresses*: murat.kirisci@istanbul.edu.tr (Murat Kirişci)**

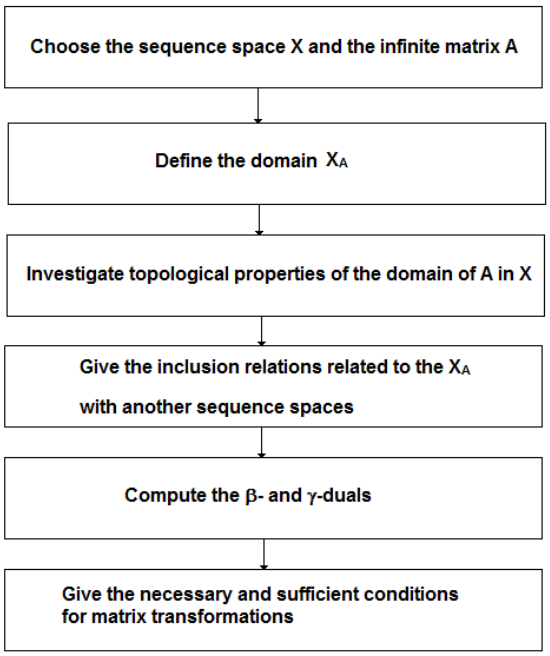

Figure 1.1: Flowchart of constructing a new sequence space

for each $n \in \mathbb{N}$, provided the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$. Let X and Y be two sequence spaces. If Ax exists and is in Y for every sequence $x = (x_k) \in X$, then we say that A defines a matrix mapping from X into Y, and we denote it by writing $A : X \to Y$ if and only if the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y$ for all $x \in X$. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x. Let X be a sequence space and A be an infinite matrix. The sequence space

$$X_A = \{x = (x_k) \in \omega : Ax \in X\} \tag{1.2}$$

is called the domain of A in X which is a sequence space.

The matrix $\Omega = (a_{nk})$ defined by $a_{nk} = k$, $(1 \le k \le n)$ and $a_{nk} = 0$, $(k > n)$, and the matrix $\Gamma = (b_{nk})$ defined by by $b_{nk} = 1/k$, $(1 \le k \le n)$ and $b_{nk} = 0$, $(k > n)$, respectively, i.e.,

$$a_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \cdots \\ 1 & 2 & 3 & 0 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad b_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1/2 & 0 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We can give the matrices $\Omega^{-1} = (c_{nk})$ and $\Gamma^{-1} = (d_{nk})$ which are inverse of the above matrices by $c_{nk} = 1/n$, $(n = k)$, $c_{nk} = -1/n$, $(n-1 = k)$, $c_{nk} = 0$, $(other)$ and $d_{nk} = n$, $(n = k)$, $d_{nk} = -n$, $(n-1 = k)$, $d_{nk} = 0$, $(other)$, respectively, i.e.,

$$c_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & -1/3 & 1/3 & 0 & \cdots \\ 0 & 0 & -1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad d_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -2 & 2 & 0 & 0 & \cdots \\ 0 & -3 & 3 & 0 & \cdots \\ 0 & 0 & -4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Now, we show that the matrices Ω and Γ preserve the limits on the set of all convergent sequences.

Theorem 1.1. *The matrices Ω and Γ are regular.*

Proof. Take a sequence $x = (x_k)$. We must show that if for $n \to \infty$ and some L, $\lim_n |x_k - L| \to 0$, then, $\lim_n |b_{nk}x_k - L| \to 0$, where b_{nk} is Γ matrix. Suppose that for $n \to \infty$ and some L, $\lim_n |x_k - L| \to 0$, and choose $\varepsilon > 0$. Then, there exists a positive integer N such that $\lim_n |x_k - L| < \varepsilon$ for $n \ge N$. Then, for $n \ge N$ and $N \in \mathbb{N}$, $\lim_n |b_{nk}x_k - L| = \lim_n |\sum_{k=1}^{n}(k^{-1}x_k - L)| < \varepsilon$. Therefore the matrix Γ is regular.

Similarly, we can show that the matrix Ω is regular. □

The paper is organized into six sections. After the introduction in Section 1, new sequence spaces are constructed in Section 2. Also, some topological properties of these new spaces are investigated in Section 2 . Section 3 describes and computes the dual spaces. The dual spaces are very important for matrix transformations. Section 4 is dedicated to characterization of matrix mappings. Examples related to the new spaces are in Section 5. Finally, Section 6 presents the conclusion(Figure 1.1).

2. New Spaces and Topological Properties

1:	Take an infinite matrix A
2:	Apply to the sequence space X
3:	**If** the matrix A is a triangle and $f : X_A \to X$ is bijective, **then**
4:	X_A and X are linearly isomorphic
5:	Investigate the topological properties of X_A
6:	**If** A is triangle, **then**
7:	X_A is a BK-space
8:	Compute the beta- and gamma-duals
9:	**do**
10:	Characterize the matrix mappings
11:	**while**(exist beta- and gamma-duals)

Table 1: Algorithm related to the constructing a new space

Now, we introduce the new sequence spaces derived by the $\Omega-$ and $\Gamma-$ matrices as follows:

$$\ell_\infty(\Omega) = \left\{x = (x_k) \in \omega : \Omega x \in \ell_\infty\right\}$$
$$c(\Omega) = \{x = (x_k) \in \omega : \Omega x \in c\}$$
$$c_0(\Omega) = \{x = (x_k) \in \omega : \Omega x \in c_0\}$$

and

$$\ell_\infty(\Gamma) = \{x = (x_k) \in \omega : \Gamma x \in \ell_\infty\}$$
$$c(\Gamma) = \{x = (x_k) \in \omega : \Gamma x \in c\}$$
$$c_0(\Gamma) = \{x = (x_k) \in \omega : \Gamma x \in c_0\}$$

Let us define the sequences $u = (u_n)$ and $v = (v_n)$, as the $\Omega-$transform and $\Gamma-$transform of a sequence $x = (x_k)$, respectively, that is, for $k,n \in \mathbb{N}$, $u_n = (\Omega x)_n = \sum_{k=1}^{n} |kx_k|$ and $v_n = (\Gamma x)_n = \sum_{k=1}^{n} \left|k^{-1}x_k\right|$.

Theorem 2.1. *The new bounded, convergent and null sequence spaces are norm isomorphic to the classical sets consisting of the bounded, convergent and null sequences.*

Proof. We will show that there is a linear isometry between new bounded, convergent, null sequence spaces and classical bounded, convergent and null convergent sequence space. We consider the transformation defined Φ, from $X(\Omega)$ to X by $x \mapsto u = \Phi x = \sum_{k=1}^{n} |kx_k|$, where $X = \{\ell_\infty, c, c_0\}$. Then, it is clear that the equality $\Phi(a+b) = \Phi(a) + \Phi(b)$ holds. Choose $\lambda \in \mathbb{R}$. Then,

$$\Phi(\lambda a) = \Phi(\lambda a_k) \quad = \quad \sum_{k=1}^{n} |\lambda k a_k| = \lambda \sum_{k=1}^{n} |ka_k| = \lambda \Phi a.$$

Therefore, we can say that Φ is linear.

Choose a sequence $y = (y_k)$ in $X(\Omega)$ and define the sequence $x = (x_k)$ such that $x = (c_{nk}y_k)$, where c_{nk} is inverse of $\Omega = (a_{nk})$ matrix. Then,

$$\|x\|_{\ell_\infty(\Omega)} \quad = \quad \sup_k |a_{nk}x_k| = \sup_k |a_{nk}c_{nk}y_k|_{\ell_\infty} = \|y\|_{\ell_\infty}.$$

Therefore, we can say that Φ is norm preserving.

Similarly, we can also show that the other spaces are norm isomorphic to classical sequence spaces. □

Theorem 2.2. *The new bounded, convergent and null sequence spaces are $BK-$spaces with the norms defined by $\|x\|_{X(\Omega)} = \|\Omega x\|_{\ell_\infty}$ and $\|x\|_{X(\Gamma)} = \|\Gamma x\|_{\ell_\infty}$, respectively, where $X = \{\ell_\infty, c, c_0\}$.*

Proof. Take a sequence $x = (x_k)$ in $X(\Omega)$, where $X = \{\ell_\infty, c, c_0\}$ and define $f_k(x) = x_k$ for all $k \in \mathbb{N}$. Then, we have

$$\|x\|_{X(\Omega)} = \sup\{1|x_1| + 2|x_2| + 3|x_3| + \cdots + k|x_k| + \cdots\}$$

Therefore, $k|x_k| \leq \|x\|_{X(\Omega)} \Rightarrow |x_k| \leq K\|x\|_{X(\Omega)} \Rightarrow |f_k(x)| \leq K\|x\|_{X(\Omega)}$. From this result, we say that f_k is a continuous linear functional for each k. Then, $X(\Omega)$ is a $BK-$space.

In the same idea, we can prove that the space $X(\Gamma)$ is a $BK-$space. □

Remark 2.3. *We can give the proof of Theorem 2.2 in a different way: From 4.3.1 of [9], we know that if a sequence space X is $BK-$space with respective norm and A is a triangular infinite matrix, then the matrix domain X_A is also $BK-$space with respective norm.*

Theorem 2.4. *The spaces $X(\Omega)$ and $X(\Gamma)$ have $AK-$property.*

Theorem 2.5. *The spaces $X(\Omega)$ and $X(\Gamma)$ have monotone norm.*

Theorem 2.4 and 2.5 can be proved as Theorem 2.4., Theorem 2.6. of [5].

Remark 2.6. *Any space with a monotone norm has AB(10.3.12 of [9]).*

Corollary 2.7. *The spaces $X(\Omega)$ and $X(\Gamma)$ have AB.*

Theorem 2.8. *The following statements hold:*

(i) Define a sequence $t^{(k)} := \{t_n^{(k)}\}_{n\in\mathbb{N}}$ of elements of the space $X(\Omega)$ for every fixed $k \in \mathbb{N}$ by

$$t_n^{(k)} = \begin{cases} (-1)^{n-k}k^{-1} & , \quad (n-1 \leq k \leq n) \\ 0 & , \quad (1 \leq k \leq n-1) \quad or \quad (k > n) \end{cases}$$

Then the sequence $\{t^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space $X(\Omega)$ and if we choose $E_k = (\Omega x)_k$ for all $k \in \mathbb{N}$, then any $x \in X(\Omega)$ has a unique representation of the form

$$x := \sum_k E_k t^{(k)}.$$

(ii) Define a sequence $s^{(k)} := \{s_n^{(k)}\}_{n\in\mathbb{N}}$ of elements of the space $X(\Gamma)$ for every fixed $k \in \mathbb{N}$ by

$$s_n^{(k)} = \begin{cases} (-1)^{n-k}k & , \quad (n-1 \leq k \leq n) \\ 0 & , \quad (1 \leq k \leq n-1) \quad or \quad (k > n) \end{cases}$$

Then the sequence $\{s^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space $X(\Gamma)$ and if we choose $F_k = (\Gamma x)_k$ for all $k \in \mathbb{N}$, then any $x \in X(\Gamma)$ has a unique representation of the form

$$x := \sum_k F_k s^{(k)}.$$

Remark 2.9. *If a space has a Schauder basis, then it is separable.*

Corollary 2.10. *The spaces $X(\Omega)$ and $X(\Gamma)$ are separable.*

1:	Take X_A
2:	Define $f : X_A \to X$
3:	**If** f is an isomorphic and surjective, **then**
4:	the inverse image of basis of X is the basis of X_A
5:	**If** X has a Schauder basis, **then**
6:	X_A is separable

Table 2: Algorithm for basis and separability

In this section, we have defined the new spaces derived by infinite matrices and examined some structural and topological properties.

3. Dual Spaces

In this section, we compute dual spaces of new defined spaces. The beta-, gamma-duals of new defined spaces will help us in the characterization of the matrix mappings.

From Lemma 5.3 of [3] and Theorem 3.1 of [1], we will give an algorithm, which provides convenience to compute $\alpha-$, $\beta-$ and $\gamma-$ duals of these new spaces and characterize some matrix transformations.

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers. We write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ and $M(X,Y) = \bigcap_{x\in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the *multiplier space* of X and Y. In the special cases of $Y = \{\ell_1, cs, bs\}$, we write $x^{\alpha} = x^{-1} * \ell_1$, $x^{\beta} = x^{-1} * cs$, $x^{\gamma} = x^{-1} * bs$ and $X^{\alpha} = M(X,\ell_1)$, $X^{\beta} = M(X,cs)$, $X^{\gamma} = M(X,bs)$ for the $\alpha-$dual, $\beta-$dual, $\gamma-$dual of X. By $A_n = (a_{nk})_{k=0}^{\infty}$ we denote the sequence in the $n-$th row of A, and we write $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ $n = (0,1,...)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$, provided $A_n \in x^{\beta}$ for all n.

1:	Take the sequence spaces X and Y
2:	**If** the spaces X and Y are $BK-$spaces, **then**
3:	matrix transformations between X and Y are continuous
4:	Choose the triangular matrix T and an infinite matrix A
5:	**do**
6:	$A \in (X : Y_T)$
7:	**while** $TA(X : Y)$
8:	Define the matrix B which is inverse of T
9:	**If** the matrix B depending on a sequence $(a_k) \in \omega$, **then**
10:	$\beta-$dual is defined by $X_T^{\beta} = \{a = (a_k) \in \omega : B^T \in (X : c)\}$ **and**
11:	$\gamma-$dual is defined by $X_T^{\gamma} = \{a = (a_k) \in \omega : B^T \in (X : \ell_\infty)\}$

Table 3: Algorithm for dual spaces and matrix transformations

Now, we list the following useful conditions.

$$\sup_n \sum_k |a_{nk}| < \infty \tag{3.1}$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0 \tag{3.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} \quad exists \tag{3.3}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \left|\lim_{n\to\infty} a_{nk}\right| \tag{3.4}$$

$$\lim_n a_{nk} = 0 \quad \text{for all k} \tag{3.5}$$

$$\sup_m \sum_k \left|\sum_{n=0}^{m}\right| < \infty \tag{3.6}$$

$$\sum_n a_{nk} \quad is\ convergent\ for\ all\ k \tag{3.7}$$

$$\sum_n \sum_k a_{nk} \quad is\ convergent \tag{3.8}$$

$$\lim_n a_{nk} \quad \text{exists for all k} \tag{3.9}$$

$$\lim_m \sum_k \left|\sum_{n=m}^{\infty} a_{nk}\right| = 0 \tag{3.10}$$

Lemma 3.1. *For the characterization of the class $(X : Y)$ with $X = \{c_0, c, \ell_\infty\}$ and $Y = \{\ell_\infty, c, cs, bs\}$, we can give the necessary and sufficient conditions from Table* **??**, *where*

1. (3.1)	**2.** (3.1), (3.9)	**3.** (3.6)	**4.** (3.6), (3.7)
5. (3.1), (3.9), (3.3)	**6.** (3.6), (3.7), (3.8)	**7.** (3.9), (3.4)	**8.** (3.10)

To $\rightarrow$	ℓ_∞	c	bs	cs
From $\downarrow$				
c_0	**1.**	**2.**	**3.**	**4.**
c	**1.**	**5.**	**3.**	**6.**
ℓ_∞	**1.**	**7.**	**3.**	**8.**

Table 4

For using in the proof of Theorem 3.2, we define the matrices $U = (u_{nk})$ and $V = (v_{nk})$ as below:

$$u_{nk} = \begin{cases} \frac{a_k}{k} - \frac{a_{k+1}}{k+1} & , \quad (k < n) \\ \frac{a_n}{n} & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases} \tag{3.11}$$

$$v_{nk} == \begin{cases} ka_k - (k+1)a_{k+1} & , \quad (k < n) \\ na_n & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases} \tag{3.12}$$

Theorem 3.2. *The $\beta-$ and $\gamma-$ duals of the new sequence spaces defined by*

$$[c_0(\Omega)]^{\beta} = \{a = (a_k) \in \omega : U \in (c_0 : c)\}$$
$$[c(\Omega)]^{\beta} = \{a = (a_k) \in \omega : U \in (c : c)\}$$
$$[\ell_\infty(\Omega)]^{\beta} = \{a = (a_k) \in \omega : U \in (\ell_\infty : c)\}$$
$$[c_0(\Omega)]^{\gamma} = \{a = (a_k) \in \omega : U \in (c_0 : \ell_\infty)\}$$
$$[c(\Omega)]^{\gamma} = \{a = (a_k) \in \omega : U \in (c : \ell_\infty)\}$$
$$[\ell_\infty(\Omega)]^{\gamma} = \{a = (a_k) \in \omega : U \in (\ell_\infty : \ell_\infty)\}$$

Proof. We will only show the $\beta-$ and $\gamma-$ duals of the new null convergent sequence spaces. Let $a = (a_k) \in \omega$. We begin the equality

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k k^{-1}(y_k - y_{k-1}) \tag{3.13}$$

$$= \sum_{k=1}^{n-1} \left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1} \right) y_k + \frac{a_n}{n} y_n = (Uy)_n$$

where $U = (u_{nk})$ is defined by (3.11). Using (3.13), we can see that $ax = (a_k x_k) \in cs$ or bs whenever $x = (x_k) \in c_0(\Omega)$ if and only if $Uy \in c$ or ℓ_∞ whenever $y = (y_k) \in c_0$. Then, from the algorithm in Table **??** , we obtain the result that $a = (a_k) \in (c_0(\Omega))^{\beta}$ or $a = (a_k) \in (c_0(\Omega))^{\gamma}$ if and only if $U \in (c_0 : c)$ or $U \in (c_0 : \ell_\infty)$, which is what we wished to prove. □

Theorem 3.3. *The $\beta-$ and $\gamma-$ duals of the new sequence spaces defined by*

$$[c_0(\Gamma)]^{\beta} = \{a = (a_k) \in \omega : V \in (c_0 : c)\}$$
$$[c(\Gamma)]^{\beta} = \{a = (a_k) \in \omega : V \in (c : c)\}$$
$$[\ell_\infty(\Gamma)]^{\beta} = \{a = (a_k) \in \omega : V \in (\ell_\infty : c)\}$$
$$[c_0(\Gamma)]^{\gamma} = \{a = (a_k) \in \omega : V \in (c_0 : \ell_\infty)\}$$
$$[c(\Gamma)]^{\gamma} = \{a = (a_k) \in \omega : V \in (c : \ell_\infty)\}$$
$$[\ell_\infty(\Gamma)]^{\gamma} = \{a = (a_k) \in \omega : V \in (\ell_\infty : \ell_\infty)\}$$

where $V = (v_{nk})$ is defined by (3.12).

4. Matrix Mapping

Let X and Y be arbitrary subsets of ω. We shall show that, the characterizations of the classes (X, Y_T) and (X_T, Y) can be reduced to that of (X, Y), where T is a triangle.

It is well known that if $h_{c_0}(\Delta^{(m)}) \cong c_0$, then the equivalence

$$x \in h_{c_0}(\Delta^{(m)}) \Leftrightarrow y \in c_0$$

holds. Then, the following theorems will be proved and given some corollaries which can be obtained to that of Theorems 4.1 and 4.2. Then, using the algorithm in Table **??**, we have:

Theorem 4.1. *Consider the infinite matrices $A = (a_{nk})$ and $D = (d_{nk})$. These matrices get associated with each other in the following relations:*
These matr

$$d_{nk} = \frac{a_{nk}}{k} - \frac{a_{n,k+1}}{k+1} \tag{4.1}$$

for all $k, m, n \in \mathbb{N}$. Then, the following statements are true:
i. *$A \in (c_0(\Omega) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [c_0(\Omega)]^{\beta}$ for all $n \in \mathbb{N}$ and $D \in (c_0 : Y)$, where Y is any sequence space.*
ii. *$A \in (c(\Omega) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [c(\Omega)]^{\beta}$ for all $n \in \mathbb{N}$ and $D \in (c : Y)$, where Y is any sequence space.*
iii. *$A \in (\ell_\infty(\Omega) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [\ell_\infty(\Omega)]^{\beta}$ for all $n \in \mathbb{N}$ and $D \in (\ell_\infty : Y)$, where Y is any sequence space.*

Proof. We assume that the (4.1) holds between the entries of $A = (a_{nk})$ and $D = (d_{nk})$. Let us remember that from Theorem 2.1, the spaces $c_0(\Omega)$ and c_0 are linearly isomorphic. Firstly, we choose any $y = (y_k) \in c_0$ and consider $A \in (c_0(\Omega) : Y)$. Then, we obtain that $D\Omega$ exists and $\{a_{nk}\} \in (c_0(\Omega))^{\beta}$ for all $k \in \mathbb{N}$. Therefore, the necessity of (4.1) yields and $\{d_{nk}\} \in c_0^{\beta}$ for all $k, n \in \mathbb{N}$. Hence, Dy exists for each $y \in c_0$. Thus, if we take $m \to \infty$ in the equality

$$\sum_{k=1}^{m} a_{nk} x_k = \sum_{k=1}^{m} a_{nk} \left(\frac{a_{nk}}{k} - \frac{a_{n,k+1}}{k+1} \right) y_k$$

for all $m,n \in \mathbb{N}$, then, we understand that $Dy = Ax$. So, we obtain that $D \in (c_0 : Y)$.

Now, we consider that $\{a_{nk}\}_{k\in\mathbb{N}} \in (c_0(\Omega))^{\beta}$ for all $n \in \mathbb{N}$ and $D \in (c_0 : Y)$. We take any $x = (x_k) \in c_0(\Omega)$. Then, we can see that Ax exists. Therefore, from the equality

$$\sum_k d_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$, we obtain that $Ax = Dy$. Therefore, this shows that $A \in (c_0(\Omega) : Y)$. □

Theorem 4.2. *Consider that the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ with*

$$e_{nk} := \sum_{k=1}^{\infty} \sum_{j=1}^{n} a_{jk}. \tag{4.2}$$

Then, the following statements are true:
i. *$A = (a_{nk}) \in (X : c_0(\Omega))$ if and only if $E \in (X : c_0)$*
ii. *$A = (a_{nk}) \in (X : c(\Omega))$ if and only if $E \in (X : c)$*
iii. *$A = (a_{nk}) \in (X : \ell_\infty(\Omega))$ if and only if $E \in (X : \ell_\infty)$*

Proof. We take any $z = (z_k) \in X$. Using the (4.2), we have

$$\sum_{k=1}^{m} e_{nk} z_k = \sum_{k=1}^{m} \left(\sum_{k=1}^{\infty} \sum_{j=1}^{m} j b_{jk} \right) z_k \tag{4.3}$$

for all $m,n \in \mathbb{N}$. Then, for $m \to \infty$, equation (4.3) gives us that $(Ez)_n = \{\Omega(Az)\}_n$. Therefore, one can immediately observe from this that $Az \in c_0(\Omega)$ whenever $z \in X$ if and only if $Ez \in c_0$ whenever $z \in X$. Thus, the proof is completed. □

Theorem 4.3. *Consider the infinite matrices $A = (a_{nk})$ and $F = (f_{nk})$. These matrices get associated with each other in the following relations:*

$$f_{nk} = k a_{nk} - (k+1) a_{n,k+1} \tag{4.4}$$

for all $k,m,n \in \mathbb{N}$. Then, the following statements are true:
i. *$A \in (c_0(\Gamma) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [c_0(\Gamma)]^{\beta}$ for all $n \in \mathbb{N}$ and $F \in (c_0 : Y)$, where Y is any sequence space.*
ii. *$A \in (c(\Gamma) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [c(\Gamma)]^{\beta}$ for all $n \in \mathbb{N}$ and $F \in (c : Y)$, where Y is any sequence space.*
iii. *$A \in (\ell_\infty(\Gamma) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [\ell_\infty(\Gamma)]^{\beta}$ for all $n \in \mathbb{N}$ and $F \in (\ell_\infty : Y)$, where Y is any sequence space.*

Theorem 4.4. *Consider that the infinite matrices $A = (a_{nk})$ and $G = (g_{nk})$ with*

$$g_{nk} := \sum_{k=1}^{\infty} \sum_{j=1}^{n} j^{-1} a_{jk} \tag{4.5}$$

Then, the following statements are true:
i. *$A = (a_{nk}) \in (X : c_0(\Gamma))$ if and only if $G \in (X : c_0)$*
ii. *$A = (a_{nk}) \in (X : c(\Gamma))$ if and only if $G \in (X : c)$*
iii. *$A = (a_{nk}) \in (X : \ell_\infty(\Gamma))$ if and only if $G \in (X : \ell_\infty)$*

5. Examples

If we choose any sequence spaces X and Y in Theorem 4.1 and 4.2 in previous section, then, we can find several consequences in every choice. For example, if we take the space ℓ_∞ and the spaces which are isomorphic to ℓ_∞ instead of Y in Theorem 4.1, we obtain the following examples:

Example 5.1. *The Euler sequence space e_∞^r is defined by $e_\infty^r = \{x \in \omega : \sup_{n\in\mathbb{N}} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k| < \infty\}$ ([2] and [4]). We consider the infinite matrix $A = (a_{nk})$ and define the matrix $H = (h_{nk})$ by*

$$h_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j a_{jk} \qquad (k,n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(c_0(\Omega) : e_\infty^r)$ in Theorem 4.1, then, we replace the entries of the matrix A by those of the matrix H.

Example 5.2. *Let $T_n = \sum_{k=0}^{n} t_k$ and $A = (a_{nk})$ be an infinite matrix. We define the matrix $P = (p_{nk})$ by*

$$p_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk} \qquad (k,n \in \mathbb{N}).$$

Then, the necessary and sufficient conditions in order for A belongs to the class $(c_0(\Omega) : r_\infty^t)$ are obtained from in Theorem 4.1 by replacing the entries of the matrix A by those of the matrix P; where r_∞^t is the space of all sequences whose R^t-transforms is in the space ℓ_∞ [7].

Example 5.3. *In the space r^t_∞, if we take $t=e$, then, this space become to the Cesaro sequence space of non-absolute type X_∞ [8]. As a special case, Example 5.2 includes the characterization of class $((c_0(\Omega):r^t_\infty)$.*

Example 5.4. *The Taylor sequence space t^r_∞ is defined by $t^r_\infty=\{x\in\omega:\sup_{n\in\mathbb{N}}|\sum_{k=n}^{\infty}\binom{k}{n}(1-r)^{n+1}r^{k-n}x_k|<\infty\}$ ([6]). We consider the infinite matrix $A=(a_{nk})$ and define the matrix $T=(t_{nk})$ by*

$$t_{nk}=\sum_{k=n}^{\infty}\binom{k}{n}(1-r)^{n+1}r^{k-n}a_{jk}\qquad(k,n\in\mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(c_0(\Omega):t^r_\infty)$ in Theorem 4.1, then, we replace the entries of the matrix A by those of the matrix T.

If we take the spaces c, cs and bs instead of X in Theorem 4.2, or Y in Theorem 4.1 we can write the following examples. Firstly, we give some conditions and following lemmas:

$$\lim_k a_{nk}=0\quad\text{for all n ,}\tag{5.1}$$

$$\lim_{n\to\infty}\sum_k a_{nk}=0,\tag{5.2}$$

$$\lim_{n\to\infty}\sum_k |a_{nk}|=0,\tag{5.3}$$

$$\lim_{n\to\infty}\sum_k |a_{nk}-a_{n,k+1}|=0,\tag{5.4}$$

$$\sup_n\sum_k |a_{nk}-a_{n,k+1}|<\infty\tag{5.5}$$

$$\lim_k\left(a_{nk}-a_{n,k+1}\right)\ \text{exists for all k}\tag{5.6}$$

$$\lim_{n\to\infty}\sum_k |a_{nk}-a_{n,k+1}|=\sum_k\left|\lim_{n\to\infty}(a_{nk}-a_{n,k+1})\right|\tag{5.7}$$

$$\sup_n\left|\lim_k a_{nk}\right|<\infty\tag{5.8}$$

Lemma 5.5. *Consider that $X\in\{\ell_\infty,c,bs,cs\}$ and $Y\in\{c_0\}$. The necessary and sufficient conditions for $A\in(X:Y)$ can be read from the Table ??:*

9. (5.3)	**10.** (3.1), (3.5), (5.2)	**11.** (5.1), (5.4)	**12.** (3.5), (5.5)
13. (5.1), (5.6), (5.7)	**14.** (5.5), (3.9)	**15.** (5.1), (5.5)	**16.** (5.5), (5.8)

From $\rightarrow$	ℓ_∞	c	bs	cs
To $\downarrow$				
c_0	**9.**	**10.**	**11.**	**12.**
c	**7.**	**5.**	**13.**	**14.**
ℓ_∞	**1.**	**1.**	**15.**	**16.**

Table 5

Example 5.6. *We choose $X\in\{c_0(\Omega),c(\Omega),\ell_\infty(\Omega)\}$ and $Y\in\{\ell_\infty,c,cs,bs,f\}$. The necessary and sufficient conditions for $A\in(X:Y)$ can be taken from the Table ??:*

1a. (3.1) holds with d_{nk} instead of a_{nk}.
2a. (3.1), (3.9) hold with d_{nk} instead of a_{nk}.
3a. (3.6) holds with d_{nk} instead of a_{nk}.
4a. (3.6), (3.7) hold with d_{nk} instead of a_{nk}.
5a. (3.1), (3.9), (3.3) hold with d_{nk} instead of a_{nk}.
6a. (3.6), (3.7), (3.8) hold with d_{nk} instead of a_{nk}.
7a. (3.9), (3.4) hold with d_{nk} instead of a_{nk}.
8a. (3.10) holds with d_{nk} instead of a_{nk}.

Example 5.7. *Consider that the $X\in\{\ell_\infty,c,bs,cs\}$ and $Y\in\{c_0(\Omega),c(\Omega),\ell_\infty(\Omega)\}$. The necessary and sufficient conditions for $A\in(X:Y)$ can be read from the Table ??:*

9a. *(5.3) holds with e_{nk} instead of a_{nk}.*
10a. *(3.1), (3.5), (5.2) hold with e_{nk} instead of a_{nk}.*
11a. *(5.1), (5.4) hold with e_{nk} instead of a_{nk}.*
12a. *(3.5), (5.5) hold with e_{nk} instead of a_{nk}.*

To $\rightarrow$	ℓ_∞	c	bs	cs
From $\downarrow$				
$c_0(\Omega)$	**1a.**	**2a.**	**3a.**	**4a.**
$c(\Omega)$	**1a.**	**5a.**	**3a.**	**6a.**
$\ell_\infty(\Omega)$	**1a.**	**7a.**	**3a.**	**8a.**

Table 6

13a. *(5.1), (5.6), (5.7) hold with e_{nk} instead of a_{nk}.*
14a. *(5.5), (3.9) hold with e_{nk} instead of a_{nk}.*
15a. *(5.1), (5.5) hold with e_{nk} instead of a_{nk}.*
16a. *(5.5), (5.8) hold with e_{nk} instead of a_{nk}.*

From $\rightarrow$	ℓ_∞	c	bs	cs
To $\downarrow$				
$c_0(\Omega)$	**9a.**	**10a.**	**11a.**	**12a.**
$c(\Omega)$	**7a.**	**5a.**	**13a.**	**14a.**
$\ell_\infty(\Omega)$	**1a.**	**1a.**	**15a.**	**16a.**

Table 7

With the same idea of Example 5.6 and Example 5.7, we can write the examples related to the Γ matrix as table form. In examples which are writing with Γ matrix, we use the f_{nk} and g_{nk}.

6. Conclusion

We know that the most general linear operators between two sequence spaces is given by an infinite matrix. The theory of matrix transformations deals with establishing necessary and sufficient conditions on the entries of a matrix to map a sequence space X into a sequence space Y. This is a natural generalization of the problem to characterize all summability methods given by infinite matrices that preserve convergence.

In this work, we construct new sequence spaces by means of the matrix domain with two infinite matrices. We examine some properties such as isomorphism, $BK-$space, $AK-$ and $AB-$properties, monotone norm. Also, we give dual spaces and later the necessary and sufficient conditions on the matrix transformations of the classes $(X:Y)$. Afterward, in the last section, we obtain several examples related to new spaces.

In this paper, a flowchart showing the stages of the formation of a new sequence space is designed. Algorithms have been produced to construction of a new sequence space, base, separability, calculation of dual spaces and matrix characterizations.

References

[1] B. Altay and F. Başar, *Certain topological properties and duals of the domain of a triangle matrix in a sequence spaces*, J. Math. Analysis and Appl., **336** (2007), 632–645.
[2] B. Altay, F. Başar, *On some Euler sequence spaces of non-absolute type*, Ukrainian Math. J. **57**(1)(2005), 1–17.
[3] F. Başar and B. Altay *On the space of sequences of p-bounded variation and related matrix mappings*, Ukranian Math. J., **55(1)** (2003), 136–147.
[4] B. Altay, F. Başar, M. Mursaleen, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I*, Inform. Sci. **176**(10)(2006), 1450–1462.
[5] M. Kirisci, *Integrated and differentiated sequence spaces*, **2015(1)**, (2015), Article ID: jnna-00266, 2–16, doi: 10.5899/2015/jnaa-00266.
[6] M. Kirişci, *On the Taylor sequence spaces of nonabsolute type which include the spaces c_0 and c*,J. Math. Anal., **6(2)**, 22–35, (2015).
[7] E. Malkowsky, *Recent results in the theory of matrix transformations in sequence spaces*, Mat. Ves. **49**, 187–196, (1997).
[8] P.-N., Ng, P. -Y., Lee, : *Cesàro sequence spaces of non-absolute type*, Comment. Math. Prace Mat. **20(2)**, 429–433, (1978).
[9] A. Wilansky, *Summability through Functional Analysis*, North- Holland (1984).