1) Harmonik salınıcı potansiyelinde bulunan bir parçacığın durumları aşağıda verilmektedir:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(u)\, e^{-u^2/2}$$

Burada $u = \sqrt{\frac{m\omega}{\hbar}}\, x$ olmak üzere $H_n(u)$ hermit polinomlarıdır. Hermite polinomlarının $n=0,1,2,3$ için değeri aşağıda verilmektedir;

$n=0 \rightarrow H_0 = 1$ $\qquad u = \sqrt{\frac{m\omega}{\hbar}}\, x$

$n=1 \rightarrow H_1 = 2u$

$n=2 \rightarrow H_2 = 4u^2 - 2$

$n=3 \rightarrow H_3 = 8u^3 - 12u$

a) ψ_0, ψ_1 ve ψ_2'yi oluşturunuz. Normalanmış olup olmadıklarını kontrol ediniz.

b) ψ_0, ψ_1 ve ψ_2'nin grafiklerini çiziniz.

c) ψ_0, ψ_1 ve ψ_2'nin ortogonalliklerini kontrol ediniz.

a: $n=0 \Rightarrow$

$$\cdot\ \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^0 0!}} \underline{H_0(u)}\, e^{-u^2/2} \qquad H_0 = 1, \quad u = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\cdot\ \langle\psi_0|\psi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-\frac{m\omega}{\hbar}x^2} dx$$

$$I = \int_0^\infty x^{2n} e^{-x^2/a^2} dx = \sqrt{\pi}\, \frac{(2n)!}{n!} \left(\frac{a}{2}\right)^{2n+1}$$

$$\Rightarrow \langle\psi_0|\psi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} 2 \int_0^\infty e^{-\frac{m\omega}{\hbar}x^2} dx$$

$1/a^2 = \frac{m\omega}{\hbar}$, $n=0$

$$I = \sqrt{\pi}\, \frac{0!}{0!} \left(\frac{a}{2}\right)^1$$

$$I = \sqrt{\pi}\, \frac{1}{2} \sqrt{\left(\frac{m\omega}{\hbar}\right)^{-1}}$$

$$I = \sqrt{\frac{\hbar\pi}{m\omega}} \cdot \frac{1}{2}$$

$$\langle\psi_0|\psi_0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\, 2 \sqrt{\frac{\pi\hbar}{m\omega}}\, \frac{1}{2} = 1 \ //$$

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$n=1 \Rightarrow$

$$\bullet\ \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^1\, 1!}} H_1(u)\, e^{-u^2/2}$$

$$H_1(u) = 2u\ , \quad u = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$H_1(x) = 2\sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}}\, 2\sqrt{\frac{m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\bullet\ \langle\psi_1|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{2m\omega}{\hbar}\right) \int_{-\infty}^{+\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}\, dx$$

$$I = \int_0^{\infty} x^{2n} e^{-x^2/a^2} dx = \sqrt{\pi}\, \frac{(2n)!}{n!} \left(\frac{a}{2}\right)^{2n+1}$$

$$a = \sqrt{\frac{\hbar}{m\omega}}\ , \quad n = 1$$

$$\langle\psi_1|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{2m\omega}{\hbar}\right) 2\int_0^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}\, dx$$

$$I = \sqrt{\pi}\, \frac{2!}{1!} \left(\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\right)^3$$

$$\langle\psi_1|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{2m\omega}{\hbar}\right) 2 \left[\sqrt{\pi}\, \frac{1}{4} \left(\frac{\hbar}{m\omega}\right)^{3/2}\right] \Rightarrow \langle\psi_1|\psi_1\rangle = 1$$

$n=2 \Rightarrow$

$$\bullet\ \psi_2(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^2 . 2!}} H_2(u)\, e^{-u^2/2}$$

$$H_2 = 4u^2 - 2\ , \quad u = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$H_2(x) = \frac{4m\omega}{\hbar} x^2 - 2$$

$$\psi_2(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{8}} \left(\frac{4m\omega}{\hbar} x^2 - 2\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\bullet\ \langle\psi_2|\psi_2\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{1}{8} \int_{-\infty}^{+\infty} \left(\frac{4m\omega}{\hbar} x^2 - 2\right)^2 e^{-\frac{m\omega}{\hbar}x^2}\, dx$$

$\langle\psi_2|\psi_2\rangle = 1$ (Ara işlemler ödev!!!)

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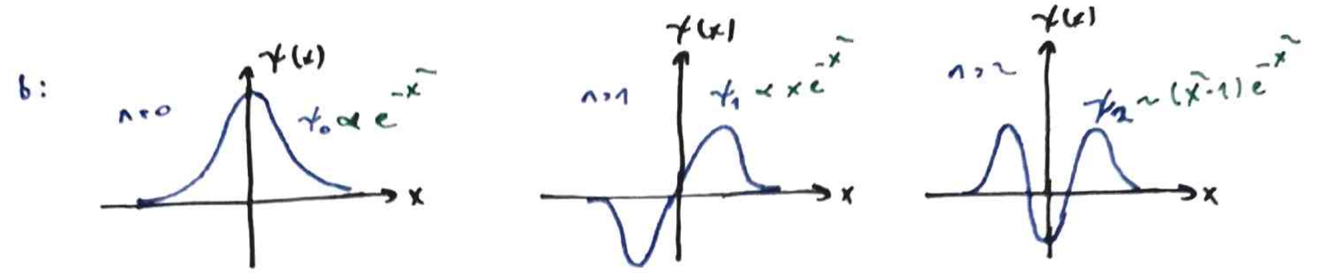
b:



c:

$$\bullet\ \langle \psi_0|\psi_1\rangle = \int_{-\infty}^{+\infty} \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}_{\psi_0} \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}}_{\psi_1} dx$$

$$\Rightarrow \langle \psi_0|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{2m\omega}{\hbar}} \int_{-\infty}^{+\infty} x\, e^{-\frac{m\omega}{\hbar}x^2} dx$$

$$\frac{m\omega}{\hbar}x^2 = u \rightarrow du = \frac{2m\omega}{\hbar} x\, dx$$

$$dx = \frac{\hbar}{2m\omega x} du$$

$$\langle \psi_0|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{2m\omega}{\hbar}} \int_{-\infty}^{+\infty} \cancel{x}\, e^{-u} \frac{\hbar}{2m\omega \cancel{x}} du$$

$$\langle \psi_0|\psi_1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{2m\omega}{\hbar}} \int_{-\infty}^{+\infty} e^{-u} du = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{2m\omega}{\hbar}} \underbrace{\left[-\frac{1}{e^u}\right]_{-\infty}^{+\infty}}_{-\frac{1}{\infty}+\frac{1}{\infty}=0} = 0$$

$\langle \psi_0|\psi_1\rangle = 0$ Ortogonaldirler.

$$\bullet\ \langle \psi_1|\psi_2\rangle = \int_{-\infty}^{+\infty} \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}}_{\psi_1} \underbrace{\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{8}\left(\frac{4m\omega}{\hbar}x^2 - 2\right) e^{-\frac{m\omega}{2\hbar}x^2}\right]}_{\psi_2} dx$$

$$\langle \psi_1|\psi_2\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\,\frac{1}{8} \int_{-\infty}^{+\infty} x\left(\frac{4m\omega}{\hbar}x^2 - 2\right) e^{-\frac{m\omega}{\hbar}x^2} dx$$

$$\langle \psi_1|\psi_2\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\,\frac{1}{8}\left[\underbrace{\int_{-\infty}^{+\infty} \frac{4m\omega}{\hbar} \underset{\text{tek fonk}}{x^3}\ \overset{\text{çift fonk}}{e^{-\frac{m\omega}{\hbar}x^2}} dx}_{0} - 2\underbrace{\int_{-\infty}^{+\infty} x\, e^{-\frac{m\omega}{\hbar}x^2} dx}_{0}\right] = 0$$

$\langle \psi_1|\psi_2\rangle = 0$ // Ortogonaldirler.

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2) Harmonik salınıcı potansiyeli için, taban durum ve 1. uyarılmış durumları için potansiyelin beklenen değerini $(\langle V\rangle)$ bulunuz.

$$V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

$\psi_0 \rightarrow$ taban durum

$\psi_1 \rightarrow$ 1. uyarılmış durum

$$\underline{\underline{\langle V\rangle}} = \frac{1}{2}m\omega^2 \underline{\underline{\langle x^2\rangle}}$$

• ψ_0 için;

$$\langle V\rangle_{\psi_0} = \langle\psi_1|V|\psi_1\rangle = \int_{-\infty}^{+\infty} \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}_{\langle\psi_0|} \underbrace{\left(\frac{1}{2}m\omega^2x^2\right)}_{V(x)} \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}_{|\psi_0\rangle} dx$$

$$\langle V\rangle_{\psi_0} = \sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{1}{2}m\omega^2 \int_{-\infty}^{+\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx$$

$n=1,\quad a=\sqrt{\frac{\hbar}{m\omega}}$

$$I = \int_0^{\infty} x^{2n} e^{-x^2/a^2} dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$$

$$I = \sqrt{\pi}\left(\frac{2!}{1!}\right)\left(\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\right)^{2\cdot1+1} = \frac{\sqrt{\pi}}{4}\left(\frac{\hbar}{m\omega}\right)^{3/2}$$

$$\langle V\rangle_{\psi_0} = \sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{1}{2}m\omega^2\, 2\underbrace{\int_0^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx}_{I=\frac{\sqrt{\pi}}{4}\left(\frac{\hbar}{m\omega}\right)^{3/2}}$$

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega$$

$$E_0 = \frac{\hbar\omega}{2}$$

$$\langle V\rangle_{\psi_0} = \sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{1}{2}m\omega^2\,\frac{\sqrt{\pi}}{2}\,\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}} = \frac{\hbar\omega}{4}$$

$$\langle V\rangle_{\psi_0} = \frac{\hbar\omega}{4} = \frac{E_0}{2} \;//$$

• ψ_1 için;

$$\langle V\rangle_{\psi_1} = \int_{-\infty}^{+\infty} \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x\,e^{-\frac{m\omega}{2\hbar}x^2}}_{\langle\psi_1|} \underbrace{\frac{1}{2}m\omega^2x^2}_{V} \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x\,e^{-\frac{m\omega}{2\hbar}x^2}}_{|\psi_1\rangle} dx$$

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$$\langle V\rangle_{\psi_1} = \sqrt{\frac{m\omega}{\pi\hbar}}\left(\frac{2m\omega}{\hbar}\right)\int_{-\infty}^{+\infty} x^4 e^{-\frac{m\omega}{\hbar}x^2}\,dx\left(\frac{1}{2}m\omega^2\right)$$

$n=2$ $\quad$ $a=\sqrt{\frac{\hbar}{m\omega}}$

$$I = \sqrt{\pi}\,\frac{(2.2)!}{2!}\left(\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\right)^5 = 12\sqrt{\pi}\,\frac{1}{32}\left(\frac{\hbar}{m\omega}\right)^{5/2}$$

$$\langle V\rangle_{\psi_1} = \sqrt{\frac{m\omega}{\pi\hbar}}\left(\frac{2m\omega}{\hbar}\right)\frac{1}{2}m\omega^2\, 2\int_{0}^{\infty} x^4 e^{-\frac{m\omega}{\hbar}x^2}\,dx$$

$$I = \frac{3\sqrt{\pi}}{8}\left(\frac{\hbar}{m\omega}\right)^{5/2}$$

$$\langle V\rangle_{\psi_1} = \sqrt{\frac{m\omega}{\pi\hbar}}\left(\frac{2m\omega}{\hbar}\right)m\omega^2\,\frac{3}{8}\sqrt{\pi}\left(\frac{\hbar}{m\omega}\right)^{5/2} = \frac{3}{4}\hbar\omega \qquad E_1 = \frac{3}{2}\hbar\omega$$

$$\langle V\rangle_{\psi_1} = \frac{3}{4}\hbar\omega = \frac{E_1}{2}\ //$$

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3) Harmonik salınıcı potansiyeli içindeki bir parçacığın başlangıç durum fonksiyonu aşağıdaki gibidir.

$$\psi(x,0) = A\left[3\,\psi_0(x,0) + 4\,\psi_1(x,0)\right]$$

a) A'yı bulunuz.

b) Parçacığın enerjisi ölçülürse hangi değerler hangi olasılıkla elde edilir?

c) $\langle E \rangle = ?$

d) $\psi(x,t)$ ve $|\psi(x,t)|^2$'yi oluşturunuz.

e) $\langle x \rangle$ ve $\langle p \rangle$'yi bulunuz.

a: $$\langle \psi|\psi\rangle = 1 = |A|^2\left[3^2 \underbrace{\langle\psi_0|\psi_0\rangle}_{\delta_{00}=1} + 4^2\underbrace{\langle\psi_1|\psi_1\rangle}_{\delta_{11}=1} + 3.4\,\underbrace{\cancel{\langle\psi_0|\psi_1\rangle}}_{\delta_{01}=0} + 4.3\underbrace{\cancel{\langle\psi_1|\psi_0\rangle}}_{\delta_{10}=0}\right]$$

$$\Rightarrow \frac{1}{|A|^2} = 9+16 \Rightarrow A = \frac{1}{5} //$$

$E_n = \left(n+\frac{1}{2}\right)\hbar\omega$

$$\psi(x,0) = \frac{3}{5}\psi_0(x,0) + \frac{4}{5}\psi_1(x,0)$$

$E_0 = \frac{\hbar\omega}{2}, \quad E_1 = \frac{3\hbar\omega}{2}$

b: $$P(E_0) = |\langle\psi_0|\psi\rangle|^2 = \left|\langle\psi_0|\left(\frac{3}{5}|\psi_0\rangle + \frac{4}{5}|\psi_1\rangle\right)\right|^2$$

$$= \left|\frac{3}{5}\underbrace{\langle\psi_0|\psi_0\rangle}_{1} + \frac{4}{5}\cancelto{0}{\langle\psi_0|\psi_1\rangle}\right|^2$$

$$P(E_0) = \frac{9}{25} //$$

$$P(E_1) = |\langle\psi_1|\psi\rangle|^2 = \left|\frac{4}{5}\right|^2 = \frac{16}{25} //$$

c: $$\langle E\rangle = \sum_{n=0}^{2} P_n(E_n)E_n = \frac{9}{25}E_0 + \frac{16}{25}E_1 = \hbar\omega\left(\frac{9}{25}\frac{1}{2} + \frac{16}{25}\frac{3}{2}\right)$$

$$\langle E\rangle = \frac{57}{50}\hbar\omega //$$

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d:

- $\Psi(x,t) = \sum_{n=0}^{2} c_n \Psi_n(x,0)\, e^{-iE_n t/\hbar}$

$$\Psi(x,t) = \frac{3}{5}\Psi_0(x,0)\, e^{-iE_0 t/\hbar} + \frac{4}{5}\Psi_1(x,0)\, e^{-iE_1 t/\hbar}$$

$$\Psi(x,t) = \frac{3}{5}\Psi_0(x,0)\, e^{-i\frac{\omega t}{2}} + \frac{4}{5}\Psi_1(x,0)\, e^{-i\frac{3}{2}\omega t} \quad //$$

- $|\Psi(x,t)|^2 = \frac{9}{25}|\Psi_0|^2 e^0 + \frac{16}{25}|\Psi_1|^2 e^0 + \frac{12}{25}\Psi_0^*\Psi_1\, e^{+i\frac{\omega t}{2}} e^{-i\frac{3}{2}\omega t} + \frac{12}{25}\Psi_1^*\Psi_0\, e^{-i\frac{\omega t}{2}} e^{+i\frac{3}{2}\omega t}$

$$\Rightarrow |\Psi(x,t)|^2 = \frac{9}{25}|\Psi_0|^2 + \frac{16}{25}|\Psi_1|^2 + \frac{12}{25}\Psi_0\Psi_1\, e^{-i\omega t} + \frac{12}{25}\Psi_1\Psi_0\, e^{+i\omega t}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$|\Psi(x,t)|^2 = \frac{9}{25}|\Psi_0|^2 + \frac{16}{25}|\Psi_1|^2 + \frac{24}{25}\Psi_0\Psi_1 \cos\omega t \quad //$$

e:

$$\langle x\rangle = \langle\Psi(x,t)|x|\Psi(x,t)\rangle = \int_{-\infty}^{+\infty} \cancel{\frac{9}{25}|\Psi_0|^2 x\, dx}^{\,0}_{\langle x\rangle_{\Psi_0}} + \frac{16}{25}\int_{-\infty}^{+\infty} \cancel{|\Psi_1|^2 x\, dx}^{\,0\ \langle x\rangle_{\Psi_1}} + \frac{24}{25}\int_{-\infty}^{+\infty}\Psi_0\Psi_1\, x \cos\omega t\, dx$$

$$\langle x\rangle = \frac{24}{25}\int_{-\infty}^{+\infty}\Psi_0\Psi_1\, dx\, \cos\omega t = \frac{24}{25}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{2m\omega}{\hbar}}\int \underbrace{e^{-\frac{m\omega}{2\hbar}x^2}}_{\Psi_0\text{'dan geldi}} \times \underbrace{x e^{-\frac{m\omega}{2\hbar}x^2}}_{\Psi_1\text{'den geldi}} dx \cos\omega t$$

$$\langle x\rangle = \frac{24}{25}\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\int_{-\infty}^{+\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} \cos\omega t = \frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t \quad //$$

- $\langle p\rangle = m\frac{d\langle x\rangle}{dt} \Rightarrow \langle p\rangle = -\frac{24}{25}\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t) \quad //$

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