1) a_+ ve a_- operatörlerini kullanarak ψ_1, ψ_2 durumlarını oluşturunuz.

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp ip + m\omega x\right), \quad \hat{p} = -i\hbar\frac{\partial}{\partial x}\hat{i} - i\hbar\frac{\partial}{\partial y}\hat{j} - i\hbar\frac{\partial}{\partial z}\hat{k}$$

$$\bullet\ \psi_1 = a_+\psi_0 = \underbrace{\frac{1}{\sqrt{2\hbar m\omega}}\left(-i\overset{-i\hbar\frac{\partial}{\partial x}}{p} + m\omega x\right)}_{a_+}\underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}_{\psi_0}$$

$$\psi_1 = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-i\left(-i\hbar\frac{\partial}{\partial x}e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x\, e^{-\frac{m\omega}{2\hbar}x^2}\right)\right]$$

$$\psi_1 = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\cancel{\hbar}\left(-\cancel{2}x\frac{m\omega}{\cancel{2}\cancel{\hbar}}e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x\, e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$\psi_1 = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[m\omega x\, e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x\, e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$\psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x\,e^{-\frac{m\omega}{2\hbar}x^2} \quad //$$

$$\bullet\ \psi_2 = a_+\psi_1 = \underline{a_+}\left(\underline{a_+\psi_0}\right) = a_+^2\psi_0$$

$$\psi_2 = \underbrace{\frac{1}{\sqrt{2\hbar m\omega}}\left(ip + m\omega x\right)}_{a_+}\underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x\,e^{-\frac{m\omega}{2\hbar}x^2}}_{\psi_1}$$

$$\psi_2 = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\left[-i\underset{-i\hbar\frac{\partial}{\partial x}}{p}\left(x\,e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$\psi_2 = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\left[-\hbar\left(e^{-\frac{m\omega}{2\hbar}x^2} + x\left(-2x\frac{m\omega}{2\hbar}e^{-\frac{m\omega}{2\hbar}x^2}\right)\right) + m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$\psi_2 = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2} \quad //$$

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2) Bir n durumunda bulunan harmonik salınıcı için potansiyel enerjinin beklenen değerini bulunuz.

$$V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 \longrightarrow \langle V\rangle = \langle \frac{1}{2}m\omega^2x^2\rangle$$

$$\langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \rightarrow x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)(a_+ + a_-)$$

$$x^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_-a_+ + a_+a_- + a_-^2\right)$$

$$a_-a_+ \neq a_+a_-$$

$$\Rightarrow \langle V\rangle_n = \frac{1}{2}m\omega^2\langle x^2\rangle_n = \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)\langle n|a_+^2 + a_-a_+ + a_+a_- + a_-^2|n\rangle$$

$$|n\rangle = |\psi_n\rangle, \quad |n+1\rangle = |\psi_{n+1}\rangle$$

$$\Rightarrow \langle V\rangle_n = \frac{\hbar\omega}{4}\left[\langle n|a_+^2|n\rangle + \langle n|a_-a_+|n\rangle + \langle n|a_+a_-|n\rangle + \langle n|a_-^2|n\rangle\right]$$

$$a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a_-|n\rangle = \sqrt{n}\,|n-1\rangle$$

- $\langle n|a_+^2|n\rangle = \langle n|a_+\left(a_+|n\rangle\right)$

$$= \langle n|a_+\left(\sqrt{n+1}\,|n+1\rangle\right)$$

$$= \langle n|\sqrt{n+1}\left(a_+|n+1\rangle\right)$$

$$= \langle n|\sqrt{n+1}\sqrt{n+2}\,|n+2\rangle = \sqrt{n+1}\sqrt{n+2}\,\underbrace{\langle n|n+2\rangle}_{\delta_{nn2}=0}$$

$$\langle n|a_+^2|n\rangle = 0 //$$

- $\langle n|a_-^2|n\rangle = \langle n|a_-(a_-|n\rangle) = \langle n|a_-\left(\sqrt{n}\,|n-1\rangle\right)$

$$= \sqrt{n}\,\langle n|a_-|n-1\rangle = \sqrt{n}\,\langle n|\sqrt{n-1}\,|n-2\rangle$$

$$= \sqrt{n}\sqrt{n-2}\,\overset{0}{\cancel{\langle n|n-2\rangle}}$$

$$\langle n|a_-^2|n\rangle = 0 //$$

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- $\langle n | a_- a_+ | n \rangle = \langle n | a_- (a_+ | n \rangle) = \langle n | a_- \sqrt{n+1} | n+1 \rangle$

$$= \sqrt{n+1} \langle n | a_- | n+1 \rangle = \sqrt{n+1} \langle n | \sqrt{n+1} | n \rangle$$

$$= \sqrt{n+1} \sqrt{n+1} \underbrace{\langle n | n \rangle}_{1}$$

$\langle n | a_- a_+ | n \rangle = (n+1) //$

- $\langle n | a_+ a_- | n \rangle = \langle n | a_+ (a_- | n \rangle) = \langle n | a_+ \sqrt{n} | n-1 \rangle = \sqrt{n} \langle n | a_+ | n-1 \rangle$

$$= \sqrt{n} \langle n | \sqrt{n} | n \rangle = \sqrt{n} \sqrt{n} \underbrace{\langle n | n \rangle}_{1}$$

$\langle n | a_+ a_- | n \rangle = n //$

$$\Rightarrow \langle V \rangle = \frac{\hbar\omega}{4} \left[0 + (n+1) + n + 0 \right] = \frac{\hbar\omega}{4} (2n+1) = \frac{\hbar\omega}{2} \left(n + \frac{1}{2} \right)$$

$$\langle V \rangle = \frac{\hbar\omega}{2} \left(n + \frac{1}{2} \right) = \frac{1}{2} \underbrace{\left(n + \frac{1}{2} \right) \hbar\omega}_{E_n} \Rightarrow \langle V \rangle = \frac{E_n}{2} //$$

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3) Harmonik salınıcının n. kararlı durumu için;

a) $\langle x \rangle$, $\langle x^2 \rangle$

b) $\langle p \rangle$, $\langle p^2 \rangle$

c) $\langle T \rangle$

herbirini a_+ ve a_- operatörlerini kullanarak hesaplayınız.

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a_+ + a_-\right) \longrightarrow x^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_-a_+ + a_+a_- + a_-^2\right)$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a_+ - a_-\right) \longrightarrow p^2 = -\frac{\hbar m\omega}{2}\left(a_+^2 - a_+a_- - a_+a_- + a_-^2\right)$$

a:

$$\bullet\ \langle n|x|n\rangle = \langle x\rangle_n = \langle n|\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|(a_+ + a_-)|n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|a_+|n\rangle + \langle n|a_-|n\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|\sqrt{n+1}\,|n+1\rangle + \langle n|\sqrt{n}\,|n-1\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\underbrace{\langle n|n+1\rangle}_{0} + \sqrt{n}\,\underbrace{\langle n|n-1\rangle}_{0}\right]$$

$$\langle x\rangle_n = 0 \,//$$

$$\bullet\ \langle x^2\rangle_n = \langle n|x^2|n\rangle = \langle n|\left[\frac{\hbar}{2m\omega}\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right)\right]|n\rangle$$

$$= \frac{\hbar}{2m\omega}\left[\underbrace{\langle n|a_+^2|n\rangle}_{\sqrt{n+1}\sqrt{n+2}\,\langle n|n+2\rangle = 0} + \langle n|a_+a_-|n\rangle + \langle n|a_-a_+|n\rangle + \underbrace{\langle n|a_-^2|n\rangle}_{\sqrt{n}\sqrt{n-1}\,\langle n|n-2\rangle = 0}\right]$$

$$= \frac{\hbar}{2m\omega}\left[\underbrace{\langle n|a_+a_-|n\rangle}_{\sqrt{n}\sqrt{n}\,\underbrace{\langle n|n\rangle}_{1}} + \underbrace{\langle n|a_-a_+|n\rangle}_{\sqrt{n+1}\sqrt{n+1}\,\underbrace{\langle n|n\rangle}_{1}}\right]$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left(n + (n+1)\right) \longrightarrow \langle x^2\rangle_n = \frac{\hbar}{2m\omega}(2n+1) \,// = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

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b)

- $\langle p \rangle_n = \langle n|p|n\rangle = \langle n|\, i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-)\,|n\rangle$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\Big[\underbrace{\langle n|a_+|n\rangle}_{\sqrt{n+1}\,\cancel{\langle n|n+1\rangle}\ 0} - \underbrace{\langle n|a_-|n\rangle}_{\sqrt{n}\,\cancel{\langle n|n-1\rangle}\ 0}\Big] = 0$$

$\langle p \rangle_n = 0$ //

veya; $\langle p \rangle = m\frac{d}{dt}\langle x\rangle \Rightarrow \langle p\rangle = 0$ //

$\langle x\rangle = 0$

- $\langle p^2 \rangle_n = \langle n|p^2|n\rangle = \langle n|\Big[-\frac{\hbar m\omega}{2}\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right)\Big]|n\rangle$

$$= -\frac{\hbar m\omega}{2}\Big[\underbrace{\cancel{\langle n|a_+^2|n\rangle}}_{0} - \underbrace{\langle n|a_+a_-|n\rangle}_{\sqrt{n}\sqrt{n}\,\underbrace{\langle n|n\rangle}_{1}} - \underbrace{\langle n|a_-a_+|n\rangle}_{\sqrt{n+1}\sqrt{n+1}\,\underbrace{\langle n|n\rangle}_{1}} + \underbrace{\cancel{\langle n|a_-^2|n\rangle}}_{0}\Big]$$

$$= -\frac{\hbar m\omega}{2}\Big[-n-(n+1)\Big] = \frac{\hbar\omega m}{2}(2n+1)$$

$\langle p^2 \rangle_n = \hbar m\omega\left(n+\frac{1}{2}\right)$ //

c) $\langle T \rangle_n = \left\langle \frac{p^2}{2m}\right\rangle_n = \frac{1}{2m}\langle p^2\rangle_n = \frac{1}{2m}\left(\hbar m\omega\left(n+\frac{1}{2}\right)\right) = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right)$

$\underline{\langle T\rangle} = \underline{\langle V\rangle} = \frac{E_n}{2} = \frac{\hbar\omega}{2}(n+1/2)$ //

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