ANALYSİS I Homework 4 12.12.2015

1. Let $f : A \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. Prove that:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a} \left| f(x) - L \right| = 0$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. Prove that:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to 0} f(x+a) = L$$

3. Let $f : (0,a) \to \mathbb{R}$, $f(x) = x^2$.

(a) For any points $x, b \in (0,a)$, show that: $\left| f(x) - b^2 \right| \le 2a|x-b|$.

(b) Use inequality in **(a)** to prove that : $\lim_{x \to b} f(x) = b^2$ for any $b \in (0,a)$

4. Let $f : (a,b) \to \mathbb{R}$ be a function and $c \in (a,b)$. Suppose there exist constants M and L such that $\left| f(x) - L \right| \le M|x-c|$ for $x \in (a,b)$. Prove that : $\lim_{x \to c} f(x) = L$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x \to 0} f(x) = L$. For $a > 0$,

Prove that : $\lim_{x \to 0} f(ax) = L$

6. Let $a \in \mathbb{R}$ and Let $f : \mathbb{R} \to \mathbb{R}$ such that .

(a) If $L = 0$, than show that $\lim_{x \to a} f(x) =$ (

(b) Give an example that if $L \ne 0$ than f may not have limit at a point a.

7. Let $A \subseteq B \subseteq \mathbb{R}$, let $f : B \to \mathbb{R}$ and let g be the restriction of f to A (That means: $g(x) = f(x)$ for all $x \in A$) Then,

(a) If f is continuous at $a \in A$, then g is continuous at $a \in A$

(b) Give an example that if g is continuous at $a \in A$, it need not follow that f is continuous at $a \in A$

8. Let $a < b < c$, Suppose that f is continuous on $[a,b]$, that g is continuous on $[b,c]$.

Define h on $[a,c]$ by $h(x) = \begin{cases} f(x), & if \;\; x \in [a,b] \\ g(x), & if \;\; x \in [b,c] \end{cases}$.

Prove that h is continuous on $[a,c]$.

9. Let $K>0$ and let $f:\mathbb{R}\to\mathbb{R}$ satisfy the condition $|f(x)-f(y)|\le K|x-y|$ for all $x, y\in\mathbb{R}$. Prove that f is continuous at every point $x_0\in\mathbb{R}$.

10. Let $f,g:A\to\mathbb{R}$ be functions and $a\in\mathbb{R}$. Suppose that $f(x)\le g(x)$ for all $x\in A$.

(a) If $\lim\limits_{x\to a} f(x)=+\infty$, then $\lim\limits_{x\to a} g(x)=+\infty$

(b) If $\lim\limits_{x\to a} g(x)=-\infty$, then $\lim\limits_{x\to a} f(x)=-\infty$

11. Let $f:A\to\mathbb{R}$ be functions such that $\lim\limits_{x\to 0} f(x)=L$. For $a>0$,

Prove that : $\lim\limits_{x\to 0} f(ax)=L$

12. Let $f:(0,\infty)\to\mathbb{R}$ be a function. Prove that :

$$\lim_{x\to+\infty} f(x)=L \quad\Leftrightarrow\quad \lim_{x\to 0^+} f\left(\frac{1}{x}\right)=L$$

13. Let $f:(a,\infty)\to\mathbb{R}$ be a function such that $\lim\limits_{x\to+\infty} xf(x)=L\in\mathbb{R}$.

Prove that : $\lim\limits_{x\to+\infty} f(x)=0$

14. Suppose that $\lim\limits_{x\to a} f(x)=L\in\mathbb{R}^+$ and that $\lim\limits_{x\to a} g(x)=+\infty$. Show that:

(a) $\lim\limits_{x\to a} f(x)g(x)=+\infty.$

(b) If $L=0$, show by an example that may fail.

$\left(i.e.\ \ if\ \ L=0, then\ give\ an\ example\ that\ \lim\limits_{x\to a} f(x)g(x)\ne+\infty.\right)$

15. 2014-2015 Öğretim yılında Final ve Telafide çıkmış sorular.

1. $a_n=(-1)^n+\left(\frac{n+1}{n+3}\right)Sin\left(\frac{n\pi}{2}\right)$ is given. $(\boldsymbol{a})$ Find all accumulation points of a_n

$(\boldsymbol{b})$ Find $\liminf\limits_{n\to\infty}(a_n)=?$ and $\limsup\limits_{n\to\infty}(a_n)=?$

2. Evaluate: $\lim_{x\to1}\frac{\sqrt{x}-1}{\sqrt{x}-x^2}=?$ $\lim_{x\to\pi}\frac{\pi-x}{Cos\left(\frac{x}{2}\right)}=?$ (Don’t use L’Hospital)

3. Show that $\lim_{x\to\sqrt{5}}(5-x^2)=0$ by using definition of limit.

4. $\forall n \in \mathbb{N}$ $0 < a_n < 1$. If the sequence (a_n) is monotone then show that the sequence

$\left(\frac{a_n}{a_n - 1}\right)$ is monotone.

5. Let $f(x)$ be an odd function. Determine whether $g(x) = f\left(\sqrt[5]{x^3 - x} - \sqrt[5]{x - x^3}\right)$ is

odd or not?

6. $f: [-1,2] \to \mathbb{R}$ is given by $f(x) = x^4 - 4sin\left(\frac{\pi x}{2}\right)$. Is there at least one x_0 in $(-1,2)$ such that $f(x_0) = 10$? Explain your answer.

7. Evaluate: $\lim_{x\to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1} =?$ $\lim_{x\to \pi} \frac{\pi - x}{sinx} =?$ (Don't use L'Hospital)

8. Show that $\lim_{x\to 2} x^2 - 4 = 0$ by using definition of limit.

9. $\forall n \in \mathbb{N}$ $0 < a_n < 1$. If the sequence (a_n) is monotone then show that the sequence

$\left(\frac{a_n}{1 - a_n}\right)$ is monotone.

10 $f(x) = \begin{cases} x^2 - 10, & |x| \le 3 \\ x + 4, & |x| > 3 \end{cases}$ is given. Is f continuous at $x = \pm 3$? If not, determine the kind of discontinuity there.

11. Let $f(x)$ be an odd function. Determine whether . $g(x) = f\left(\sqrt[5]{x^3 - x} - \sqrt[5]{x - x^3}\right)$ is odd or not?

1. $a_n = \left(3 - \frac{(-1)^n}{n}\right) cos\left(\frac{n\pi}{2}\right)$ is given. $(\boldsymbol{a})$ Find all accumulation points of a_n

$(\boldsymbol{b})$ Find $\liminf_{n\to\infty} (a_n) =?$ and $\limsup_{n\to\infty} (a_n) =?$

2. Evaluate: $(\boldsymbol{a})$ $\lim_{x\to 2} \frac{\sqrt{x+2} - 2}{x^2 - 4} =?$ $(\boldsymbol{b})$ $\lim_{x\to 3} \frac{3 - x}{sin(\pi x)} =?$ (Don't use L'Hospital)

14. Show that $\lim_{x\to 1} (1 - x^2) = 0$ by using definition of limit.

15. $\forall n \in \mathbb{N}$ $a_n > 0$. If the sequence (a_n) is monotone, then show that the sequence

$\left(\frac{1 - a_n}{1 + a_n}\right)$ is monotone.

16. Determine whether the function $f(x) = \begin{cases} -x^6, & if\ x \ge 0 \\ x^6, & if\ x < 0 \end{cases}$ is even, odd or neither.