ANALYSIS II Homework 2 17.02.2016

1. Suppose that $f:[0,2]\to\mathbb{R}$ is continuous on $[0,2]$, differentiable on $(0,2)$ and $f(0)=0,\ f(1)=1=f(2)$.

(a) Show that there exists $x_0\in(0,2)$ such that $f'(x_0)=\frac{1}{3}$

(b) Show that there exists $x_1\in(0,1)$ such that $f'(x_1)=1$

(c) Show that there exists $x_2\in(1,2)$ such that $f'(x_2)=0$

2. Let f and g be differentiable functions on $\mathbb{R}$. Suppose that $f(0)=g(0)$ and $f'(x)\le g'(x)$ for all $x\ge 0$. Show that $f(x)\le g(x)$ for all

3. Let $f:[0,\infty)\to\mathbb{R}$ be differentiable on $(0,\infty)$. Suppose that $\lim\limits_{x\to\infty} f'(x)=\beta$.

(a) Show that $\lim\limits_{x\to\infty}\frac{f(x+h)-f(x)}{h}=\beta$ for any $h>0$.

(b) Show that if $\lim\limits_{x\to\infty} f(x)=\alpha$, then $\beta=0$.

(c) Show that $\lim\limits_{x\to\infty}\frac{f(x)}{x}=\beta$ for any $h>0$.

4. Use the Mean value theorem MVT to show the following:

(a) $|\sin x-\sin y|\le|x-y|$

(b) $x\le\arcsin x\le\frac{x}{\sqrt{1-x^2}}$. When does equality hold?

(c) If $y<x$, then $\frac{x-y}{1+x^2}<|\arctan x-\arctan y|<\frac{x-y}{1+y^2}$

(d) If $0<a<x$, then $\frac{x-a}{x}<\ln x-\ln a<\frac{x-a}{a}$

5. $f(x)=\begin{cases}x^3 & ,if\ x\le 1\\ 3x-2 & ,if\ x>1\end{cases}$ is given. Does f have continuous derivative?

6. $f(x)=\begin{cases}x^2\sin\frac{1}{x^2} & ,if\ x\ne 0\\ 0 & ,if\ x=0\end{cases}$ is given. Sohw that this is an example of a function that is everywhere differentiable, but that has unbounded derivative at $x=0$.

7. $f(x)=\begin{cases} |x|^{\alpha}\sin\frac{1}{x} & ,if\ \ x\neq 0 \\ 0 & ,if\ \ x=0 \end{cases}$ is given.

For what values of α is *f* differentiable at $x=0$.

8. Show that $3x^4+4x^3+\varepsilon=0$ can have at most one real root less than or equal to -1, no matter what the value of ε.

9. Show that if f is differentiable in $\{0\leq x<a\}$, $f(0)=0$, $f'(x)>0$ and increasing, then $\frac{f(x)}{x}$ is increasing.

10. If f is differentiable at x_0, then compute:

$$\lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0+\beta h)}{h}=?$$

11. If f is defined in a neighborhood of x_0, f' is continuous there, and f'' exixsts, Show that

$$\lim_{h\to 0}\frac{f(x_0+2h)-2f(x_0+h)+f(x_0)}{h^2}=f''(x_0).$$

12. Compute the following limits by using L'Hospital Rule.

(a) $\lim_{x\to 0}\frac{e^{(x^2)}-e^4}{\tan(x^2-4)}=?$

(b) $\lim_{x\to 0}\frac{e^{(ax^2)}-e^{(bx^2)}}{x\ln(x+1)}=?$

(c) $\lim_{x\to 0}\frac{e^{(xa^x)}-1}{xa^x}=?(a>0)$

(d) $\lim_{x\to\infty}x\ln\left(\frac{x+a}{x-a}\right)=?(a\neq 0)$

(e) $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)=?$

(f) $\lim_{x\to\infty}\frac{\ln x}{x^a}=?(a>0)$

(g) $\lim_{x\to 0}\frac{1-\cos 2x-2x^2}{x^4}=?$

(h) $\lim_{x\to\infty}\frac{\arcsin x-x}{\arctan x-x}=?$

(i) $\lim_{x\to 0}(\cos x)^{1/x^2}=?$

(j) $\lim_{x\to\infty}x\left(p^{1/x}-1\right)=?(p>0)$

(k) $\lim_{x\to 0^+}(\sec 2x)^{x\sqrt{x}}=?$

(l) $\lim_{x\to 0^+}\left(x^{(x^x)}-1\right)=?$