ANALYSİS I Homework 1 21.10.2015

1. Prove the following by induction.

(a) $\forall n \in \mathbb{N};\ 1^2+3^2+5^2+\cdots+(2n-1)^2=\dfrac{n(4n^2-1)}{3}$

(b) $\forall n \in \mathbb{N};\ 1^3+3^3+5^3+\cdots+(2n-1)^3=n^2(2n^2-1)$

(c) $\forall n \in \mathbb{N}:\ 4^n-1$ is divisible by 3.

(d) $\forall n \in \mathbb{N}:\ 12^{n-1}+10$ is divisible by 11.

(e) $\forall n \in \mathbb{N}$ için $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

2. **(a)** Let A *and* B be two sets. Show that: $A \subseteq B \Leftrightarrow A \cap B = A$

(b) Let A *and* B be two sets. Show that: $A \subseteq B \Leftrightarrow A \cup B = B$

3. Find the solution sets of the following.

(a) $|x+4|+|x-3|=7$ **(b)** $|x+4|+|x-3|=7$

(c) $|x+4|-|x+3| \geq 2$ **(d)** $\dfrac{|x-3|+|x+3|}{|2-x|+1}=2$

(e) $|x+6|-|x-3| \leq 2+x$ **(f)** $|x-2|+|x-3| \geq 5-x$

4. $x, y, z \in \mathbb{R}$. Prove the followings:

(a) $\max\{x,y\}=\dfrac{x+y+|x-y|}{2}$ **(b)** $\min\{x,y\}=\dfrac{x+y-|x-y|}{2}$

(c) $\dfrac{|x+y|}{1+|x+y|} \leq \dfrac{|x|}{1+|x|}+\dfrac{|y|}{1+|y|}$ **(d)** $\min\{x,y,z\}=\min\{\min\{x,y\},z\}$

5. Let x be a positive irratioal number. Show that $\sqrt{x}$ is an irrational number.

6. Given $x \in \mathbb{Q}^+$ and $\sqrt{y} \in \mathbb{R}-\mathbb{Q}$. Is $\sqrt{x+\sqrt{y}}$ an irratioal number or not?. Why?

7. Given $x \in \mathbb{Q}$ and $y \in \mathbb{R}-\mathbb{Q}$. Is $\sqrt[3]{x.\sqrt[5]{y}}$ an irratioal number or not?. Why?

8. Given $x \in \mathbb{Q}$ and $y \in \mathbb{R}-\mathbb{Q}$. Is $\sqrt[5]{x^3+\sqrt[7]{y}}$ an irratioal number or not?. Why?

9. Given $x^2 \in \mathbb{R}-\mathbb{Q}$. Is $\sqrt{x}$ an irratioal number or not?. Why?

10. Let the A be bounded above. $(A \subseteq \mathbb{R})$

(a) For a fixed $x_0 \in \mathbb{R}$, Show that the set $x_0 + A = \{x_0 + a : a \in A\}$ is bounded above.

(b) If $SupA = \propto$, then show that $Sup(x_0 + A) = x_0 + \propto$.

(c) For a fixed $y_0 \in \mathbb{R}$, Show that the set $A - y_0 = \{a - y_0 : a \in A\}$ is bounded above.

(d) If $SupA = \propto$, then show that $Sup(A - y_0) = \propto - y_0$.

11. **(a)** For the set $A = [3,5] \subset \mathbb{R}$, show that $supA = 5,\ and\ \ infA = 3$

(b) For the set $A = [3,5) \subset \mathbb{R}$, show that $supA = 5,\ and\ \ infA = 3$

(c) For the set $A = \{\ \} \forall n \in \mathbb{N};\ 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$

12. **(a)** Let A be bounded below. Define: $-A = \{-a \in \mathbb{R} :\ a \in A\}$.

Show that the set $-A$ is bounded above.

(b) Let A be bounded above. Define: $-A = \{-a \in \mathbb{R} :\ a \in A\}$.

Show that the set $-A$ is bounded below.

(c) Define: $-A = \{-a \in \mathbb{R} :\ a \in A\}$.

Show that $sup(-A) = -infA$ and $inf(-A) = -supA$.

(d) The set $A = \left\{ \dfrac{(-1)^n}{n} : n \in \mathbb{N} \right\}$ is given. Is the set A bounded above and below?

If so find $\inf A = ?$ *and* $\sup A = ?$

(e) The set $A = \left\{ 1 - \dfrac{(-1)^n}{n} : n \in \mathbb{N} \right\}$ is given. Is the set A bounded above and below?

If so find $\inf A = ?$ *and* $\sup A = ?$

13. Let A be bounded below. Define $B = \{b \in \mathbb{R} :\ b\ \text{is a lower bound for}\ A\}$.

Show that $infA = supB$.

14. Let $a, b \in \mathbb{R}$. Then $a \leq b$ if and only if for every $n \in \mathbb{N}$ we have $a - \frac{1}{n} < b$

15. Let A *and* B be nonemty subsets of $\mathbb{R}$.

(a) Prove that : if $A \subseteq B$, then $\inf B \leq inf A \leq sup A \leq sup B$.

(b) Prove that : $Sup(A \cup B) = max\{supA \, , supB\}$

16. Let A *and* B be nonemty bonded subsets of $\mathbb{R}$.

(a) Is $A \cup B$ bounded or not? Why? **(b)** Is $A \cap B$ bounded or not? Why?

(c) Is $A - B$ bounded or not? Why?

17. Show that; if $x, y \in \mathbb{R}$ *with* $x \neq y$, then there exist $\varepsilon-$neighborhoods $B_\varepsilon(x)$ of x and $B_\varepsilon(y)$ of y such that $B_\varepsilon(x) \cap B_\varepsilon(y) = \varnothing$.

18. If a set $A \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of A.

19. Let A be a nonempty bounded set in $\mathbb{R}$.

(a) Let $k > 0$, let $kA = \{ka : a \in A\}$.

Prove that: $i)$ $\inf(kA) = k \inf A$ $ii)$ $\sup(kA) = k \sup A$

(b) Let $k < 0$, let $kA = \{ka : a \in A\}$.

Prove that: $i)$ $\inf(kA) = k \sup A$ $ii)$ $\sup(kA) = k \inf A$

(e) The set $A = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ is given. Is the set A bounded above and below?

If so find $\inf A = ?$ *and* $\sup A = ?$

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If so find $\inf A = ?$ *and* $\sup A = ?$